THE CAMBRIDGE COLLOQUIUM.

At the close of the first Colloquium, held under the auspices of the American Mathematical Society at Buffalo in September, 1896, it was the unanimous wish of the thirteen participants that the same plan should be adopted for our next annual summer meeting. At Toronto, in 1897, the meeting of the British Association made such an arrangement impracticable, and it was deferred to the following year. At the annual meeting in December, 1897, the Council appointed a committee of six to arrange a Colloquium in connection with the summer meeting at Boston. The committee sent out, June 20th, 1898, the following announcement to all members of the Society:

"In the week (August 22–27) following the regular session a Colloquium will be held at Harvard University, Cambridge, Mass. Two courses of six lectures each will be offered by Professor W. F. Osgood, of Harvard University, and Professor A. G. Webster, of Clark University. Titles and outlines of the courses are as follows:

"Professor Osgood:—'*On Some Methods and Problems of the General Theory of Functions.*'

"The object of the course is to discuss some of the methods that are fruitful in the study of questions in the general theory of functions, and to consider some of the problems in this field that have been the subject of investigation in recent years. As an instance of the former may be cited the application of methods due to Riemann to the establishment of two general theorems; the first, Picard's relating to the values that a single valued function assumes in the neighborhood of an essentially singular point; the second, Poincaré's, relating to the representation of any analytic function in terms of single valued functions. To the latter belongs the account which it is proposed to give of some recent researches by young French mathematicians.

"A working knowledge of the elements of the general theory of functions, including the subject of conformal mapping, will be assumed on the part of the audience.

"Professor Webster: '*The Partial Differential Equations Connected with Wave Propagation.*'

"Among the subjects treated will be some of the following: The properties of the ether.—Differential equations of the electromagnetic field.—Foundations of optics.—Propagation of waves in general.—The analytical establishment of Huy-

gens' principle.—Double refraction.—Propagation of electrical waves in wires.

"No knowledge of physics will be presupposed except the principles of dynamics and a few of the usual theorems of mathematical physics."

On Monday, August 27th, the following twenty-six persons met at Cambridge, in Room 24 of Sever Hall, Harvard University: Professor A. L. Baker, Professor Maxime Bôcher, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Professor T. S. Fiske, Professor A. B. Frizell, Dr. J. W. Glover, Miss Ida Griffiths, Professor J. G. Hagen, Professor F. H. Loud, Professor James McMahon, Professor H. P. Manning, Dr. G. A. Miller, Professor E. H. Moore, Dr. D. A. Murray, Professor B. O. Peirce, Professor James Pierpont, Mr. F. C. Radelfinger, Professor P. F. Smith, Dr. Virgil Snyder, Professor E. B. Van Vleck, Professor H. S. White, Miss E. C. Williams, Professor F. S. Woods, Professor T. W. D. Worthen.

It was agreed to have three lectures daily, at 10 and 11:30 A. M. and at 2:30 P. M., the lecturers alternating, and that each lecturer should act as presiding officer during his own sessions. This programme was suspended on Thursday, August 25th, when the Society accepted an invitation from Section A of the American Association for the Advancement of Science, to participate in their special Jubilee meeting. Professor Osgood gave the first lecture on Monday at 10 A. M., Professor Webster closing the series with his sixth lecture on Friday at 3 P. M.

By the courtesy of Professor B. O. Peirce, those in attendance were enabled to lunch together daily in the convenient home of the Colonial Club, adjoining the Yard. The Librarian of Harvard College extended to those desiring to use books all possible facilities, both in the reading-room and in the stack. The hospitality of Harvard University deserves the most grateful acknowledgment.

The members of the Colloquium, immediately before adjournment, gave formal expression to their appreciation of the value of the lectures, and of the self-forgetful public spirit of the lecturers.

Professors Webster and Osgood have consented to prepare reports of their lectures for publication in the BULLETIN, Professor Osgood's appearing in the present number.

H. S. WHITE.

SELECTED TOPICS IN THE GENERAL THEORY OF FUNCTIONS.

SIX LECTURES DELIVERED BEFORE THE CAMBRIDGE COLLOQUIUM, AUGUST 22-27, 1898.

BY PROFESSOR W. F. OSGOOD.*

Lecture I.

Picard's Theorem, and the Application of Riemann's Geometric Methods in the General Theory of Functions.

THE subject which I have chosen for the first lecture of the Colloquium is Picard's noted theorem which in its more restricted form† may be stated as follows: *Any function $G(z)$ which is single valued and analytic for all finite values of z takes on in general for at least one value of z any arbitrarily assigned value C. There may be one value, a, which the function does not take on. But if there is a second such value, b, the function reduces to a constant.*

To prove the theorem it is sufficient to establish the existence of a function $\omega(x)$ such that

(1) $\omega(x)$ is analytic for all but three values of x;

(2) $\omega(x)$ does not enter a certain region of the ω-plane, no matter what path x traces out in the x-plane.

For, let $x = G(z)$ and let the singular points of the function $\omega(x)$ be the points a, b, ∞. If z, starting with the value z_0, traces out a closed path in the z-plane, x, starting with the value x_0, will return to this value; but $\omega(x)$ may conceivably, when x describes this path, fail to return to its original value; *i. e.*, x may have described a path which *on the Riemann's surface* of the function $\omega(x)$ is not closed. To show that this is not the case, Picard reflects that, the path in the z-plane being drawn together continuously to a point, the corresponding path in the x-plane must behave likewise and hence in the course of its deformation cannot pass over any one of the points a, b, ∞. Hence $\omega(x)$, regarded as a function of z, is a single valued function, analytic for all finite values of z. Now by a well known theorem of Weier-

* To Professors E. H. Moore and H. S. White for their assistance in editing these lectures for publication I wish to express grateful acknowledgments.

† Picard, "Sur une propriété des fonctions entières," *Comptes Rendus*, vol. 88 (1879); also his Traité d'Analyse, vol. 2, p. 231.

strass's* such a function must for certain values of z come arbitrarily near to any assigned value. If we choose as this value an interior point of the region in the ω-plane that is never entered by ω, we are led to a contradiction and the theorem is proved.

Thus far we have reproduced substantially Picard's reasoning. Let us notice that the proof is intuitional in the use it makes of continuous deformations of continuous curves. To place the proof on the firmest foundation we possess it remains to arithmetize† these steps. It is entirely possible to do so, the method employed being substantially the same as that by which it is shown that the angle‡ of a continuous function of a complex variable, be the function analytic or not, comes back to its original value when z describes a closed path that forms the complete boundary of a region within and on the boundary of which the function does not vanish. I wish to call your attention, however, to a second method, also geometric, which is instructive, and which is fully as simple as the above.

Let z_0 be any value of z and let ω_0 be one of the values of $\omega(x_0)$, where $x_0 = G(z_0)$. Then, since

$$\omega - \omega_0 = \mathfrak{P}_1(x - x_0)$$

and

$$x - x_0 = \mathfrak{P}_2(z - z_0)$$

it follows that

$$\omega - \omega_0 = \mathfrak{P}(z - z_0)$$

and we wish to show that this last power series converges for *all* values of z. Suppose this were not the case. Let z_1 be one of the singular points of ω, regarded as a function of z, on the circle of convergence of $\mathfrak{P}(z - z_0)$. Connect z_0 with z_1 by a right line. As z, starting from z_0, describes this line, x describes a certain curve L on the x-surface for $\omega(x)$ and the neighborhood of each point x' of the line goes over conformally upon the neighborhood of the corresponding point

*Weierstrass, "Zur Theorie der eindeutigen analytischen Functionen," § 8; *Abh. d. Berliner Akad. d. Wiss.*, 1876.

† I do not mean to imply that no geometric proof can be as rigorous as an arithmetic one. There is every reason to expect that geometry will some time be placed on as rigorous a footing as that which arithmetic now possesses. But at the present time the most rigorous proofs which we have of theorems like the above are those which belong to that realm of geometry whose processes can be at once translated into arithmetic processes (*i. e. arithmetized*).

‡ Burkhardt uses the term *arcus* instead of the unfortunate one *amplitude* of a complex quantity, $a + bi$, to denote $\tan^{-1}\frac{b}{a}$. It may well be rendered in English by *angle*.

x' of L, or possibly, in the case of a finite number of points x_a', upon a set of $p > 1$ leaves spread out over the x-surface and connected at a branch point in x_a'. For simplicity, assume that no branch points x_a' appear. Then it is possible to mark out a (narrow) strip in the z-plane containing the line $z_0 z_1$ in its interior and related conformally to a strip in the x-plane containing the curve L in its interior. But this latter strip is related conformally to a strip in the ω-plane containing the curve corresponding to L. Thus the neighborhood of the point z_1 goes over conformally on the neighborhood of a point x_1 and the neighborhood of the latter point goes over in turn conformally on the neighborhood of a point ω_1. Hence ω, regarded as a function of z, is analytic in the point z_1, and this determination of ω in and near z_1 coincides for those values of z near z_1 which lie within the original circle about z_0 with the values given by the series $\omega - \omega_0 = \mathfrak{P}(z - z_0)$, since it can be obtained from that series by analytic continuation. This would be impossible if the point z_1 were a singular point of the function ω defined by the element

$$\omega - \omega_0 = \mathfrak{P}(z - z_0).$$

The excluded case of branch points in the points x_a' presents no difficulty.

This proof is, like the first proof, geometric. But one who is conversant with Weierstrass's theory of functions will observe that *it can be arithmetized immediately by the elementary methods of that theory.*

It may be remarked that analysis of the above nature is not infrequently useful in problems of elimination—problems too often treated superficially by formal methods.

It remains to construct the function $\omega(x)$. Picard obtained such a function by taking the ratio of the moduli of periodicity of an elliptic integral of the first kind

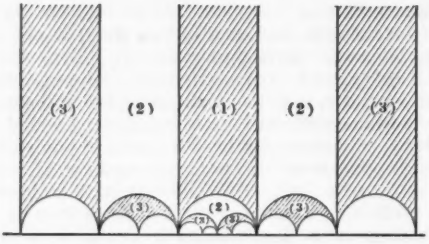
$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}},$$

$$k^2 + k'^2 = 1, \qquad \omega(k^2) = \frac{2K'i}{4K}.$$

The proof that this function has the desired properties involves some development of the theory of the elliptic transcendents. Now I wish to set over against this method for obtaining $\omega(x)$ *the general methods of Riemann's theory of functions,*—methods which may with advantage be employed

much more extensively than has hitherto been the case in the general theory of functions.

For this purpose I begin with a figure which in the geometry of inversion is a triangle each of whose angles is 0, —the shaded triangle numbered (1). Let this triangle be



reflected* on each of its sides, thus generating the non-shaded triangles numbered (2). Again, reflect each of these new triangles on each of its outer boundaries, thus generating the shaded triangles numbered (3). And so on indefinitely. A figure is thus formed with the following properties:

(1) each pair of adjacent triangles are symmetrical to each other;

(2) the triangles never enter the negative half-plane.

We are now in a position to construct the function $\omega(x)$. We may without loss of generality set $a = 0$, $b = 1$. This amounts to proving Picard's theorem for the function

$$G_1(z) = \frac{G(z) - a}{b - a},$$

whence the theorem for $G(z)$ follows. Riemann† enunci-

* By the image or reflection of a point P in a circle is meant that point P' which lies on the line OP, on the same side of O as P, and at such a distance from O that $\overline{OP}\ \ \overline{OP'} = R^2$, where R denotes the radius of the circle. A family of circles through P and P' cuts the given circle orthogonally; and conversely, if a family of circles through two fixed points cuts a given circle orthogonally, the basis-points of the family are symmetrical to each other (*i. e.*, mutual images of each other) with respect to the given circle. In any reflection of the plane, angles are preserved in magnitude, but reversed in sense, and circles go over into circles. Hence it follows that two points which are symmetrical to each other with respect to a given circle go over, when the plane is reflected in any other circle, into points symmetrical with respect to the transformed circle.

† Inaugural Dissertation, Göttingen, 1851; Gesammelte Werke, p. 40.

ated the proposition that any simply connected region can be mapped conformally on a circle and Schwarz* gave a rigorous proof of it for the case that the boundary consists of a finite number of analytic curves. Let us map then the first triangle on the positive half of the x-plane, the vertices going over into the points 0, 1, ∞. If we denote the independent variable of the first plane by ω, the map defines ω as an analytic function of x. This function can be continued across the axis of reals into the negative half-plane by Schwarz's principle of symmetry† as follows: Let P be any point of the triangle (1), Q the corresponding point in the positive half of the x-plane. Let P' be the image of P in a specified side of the shaded triangle (1) (and hence a point of a specified one of the non-shaded triangles (2)), and let Q' be the image of Q in the axis of reals. Thus the negative half-plane is mapped on the specified triangle (2). The function defined by this map is the analytic continuation of ω across that piece of the axis of reals which corresponds to the specified side of the triangle (1).

This process being repeated indefinitely, all the analytic continuations of the function $\omega(x)$ are obtained geometrically by reflections of the triangles of the ω-plane in their sides. The points 0, 1, ∞ are singular points for each branch of the function $\omega(x)$, and they are the only singular points of this function.

The function $\omega(x)$ thus defined is the function sought.

We turn now to the more general form of Picard's theorem:

If $F(z)$ is any analytic function of z which in the neighborhood of a point A is single valued and has in this region no other singularities than poles, and if A is an essentially singular point of $F(z)$, then there are at most two values which $F(z)$ does not take on in every neighborhood of the point A. If there are no poles in the neighborhood of A, there is at most one value that $F(z)$ does not take on in every neighborhood of A.

This theorem, it will be noticed, is concerned with the behavior of a function *im Kleinen*, *i. e.*, throughout a certain arbitrarily small region; while the earlier theorem was one *im Grossen*, the domain of the independent variable being there the whole finite region of the plane.

The first part of the theorem follows from the second part. For, if $F(z)$ has poles clustering about A and fails

* Cf. Picard, Traité d'Analyse, vol. 2, ch. 10.
† Ibid.

to assume the value a, then the function $\frac{1}{F(z)-a}$ has no poles in the neighborhood of z_0 and hence by the second part of the theorem can fail to assume at most one value, which surely is not 0. Denote this value by $\frac{1}{b-a}$. Then, if $c \neq a, b$, the equation

$$\frac{1}{F(z)-a} = \frac{1}{c-a}$$

can be satisfied, and hence $F(z) = c$, if z is a root of that equation.

Let then $F(z)$ be analytic everywhere in the neighborhood of the point $A = \infty$, *i. e*, in the region S lying outside of a certain large circle about $z = 0$ as center; and assume that, within S, $F(z)$ never takes on the values a, b, which we will again set equal to 0, 1. As in the earlier proof, let

$$x = F(z).$$

Then if we cut S along the positive axis of reals, $\omega(x)$, regarded as a function of z, will be single valued within this simply connected region S'. But when z crosses the cut and, after encircling the point $z = \infty$, comes back to its original value, the path of x may be an *open* path λ on the Riemann's surface for $\omega(x)$ and ω will have gone over into a new value ω_0'. Let us notice that in that case the whole neighborhood of ω_0 will have gone over conformally into the neighborhood of ω_0'. In fact, this transformation of ω, though defined originally by the paths in the x-plane and the ω-plane which correspond to the path from z_0 back to z_0, may now be studied directly by means of the last two paths alone. But here ω' appears simply as an analytic continuation of ω along a path in the x-plane. Such a continuation, we have seen, can always be obtained by reflecting the original triangle of the ω-plane successively, and since x has come back to its starting point, the reflections must be even in number; for only then will shaded regions in the ω-figure be carried over into shaded regions, non-shaded regions into non-shaded regions. And now I say: *the transformation of the region about ω_0 on the region about ω_0' is a linear transformation of the complex variable ω.* For any reflection of the plane in a circle defines a transformation of the whole plane into itself which is conformal everywhere (including the point ∞) with reversal of angles. An even number of such reflections will define, then, a transformation of the whole plane into

itself conformal everywhere *without* reversal of angles. Such a transformation defines an analytic function of a complex variable, and this function can readily be shown to be linear—a good exercise, by the way, for students in a course on the theory of functions. Hence

$$\omega' = \frac{a\omega + \beta}{\gamma\omega + \delta}.$$

Let z describe the same path again. It follows. then, by a repetition of the above reasoning that ω' will be carried over into a new branch ω'', where ω'' is a linear function of ω'. But will this be the *same* linear function as before? To answer this question we observe that the above linear relation between ω and ω' holds, when ω, ω' are regarded as functions of z, for all values in the neighborhood of z_0. It must, therefore, continue to hold for all simultaneous analytic continuations of ω and ω'. And since the function ω' can be continued analytically along the same path along which ω was just continued, it follows that at the end of the path

$$\omega'' = \frac{a\omega' + \beta}{\gamma\omega' + \delta}$$

Thus ω, regarded as a function of z, is multiple valued, taking in each point of S the values corresponding to the group of transformations consisting of the powers of the transformation

$$\omega' = \frac{a\omega + \beta}{\gamma\omega + \delta}.$$

We proceed now to form a function $\varphi(\omega)$ which shall be invariant by the transformations of this group, shall be a single valued function of z, and shall furthermore not enter every region of the φ-plane. To do this we will first show that the above linear transformation must be either

(a) the parabolic transformation

$$\frac{1}{\omega' - \omega^{(0)}} = \frac{1}{\omega - \omega^{(0)}} + k,$$

in particular $\omega' = \omega + k,$

where k is real; or

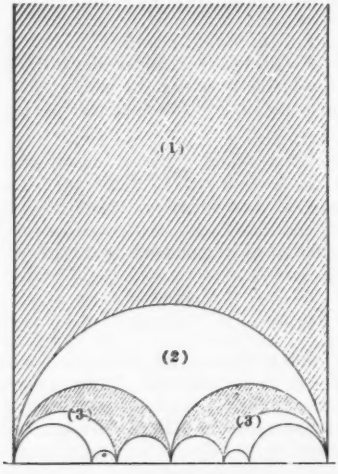
(b) the hyperbolic transformation

$$\frac{\omega' - \omega^{(1)}}{\omega' - \omega^{(0)}} = k\,\frac{\omega - \omega^{(1)}}{\omega - \omega^{(0)}},$$

where k is real, positive, and $\neq 1$.

For, since each of the reflections used in generating the automorphic figure of the ω-plane carries the upper and the lower half-planes over into themselves, the same must hold true of the linear transformation in question. Its fixed points must, therefore, either lie on the axis of reals, and in that case it will be one of the above transformations, or be symmetrically situated with reference to that axis. We proceed to show that this latter case is impossible.

The automorphic figure above generated covers the whole of the upper half-plane. For the purposes of the proof of this theorem it will be convenient to generate a new automorphic figure as follows,* and subsequently to show that it is identical with the old figure. Begin with the shaded



triangle (1) and reflect it on its semicircular side, thus generating the triangle (2). If the finite vertices of (1) were at the points 0 and 1, a vertex of (2) will be at 1/2. Now reflect the whole figure thus formed on each of the two semicircular boundaries, thus obtaining the triangles numbered (3). The maximum radius of a semicircle of the new boundary will be $1/2^2$. Next reflect the total figure thus formed in each semicircle of its boundary. The maximum radius of a semicircle of the new boundary will be $1/2^3$. And

* This method of proof was suggested to me by Professor Bôcher.

so on. Hence the boundary of the figure will ultimately sweep over any preassigned point of the upper half-plane which lies between the parallels through 0 and 1. Finally, complete the automorphic figure by an indefinite repetition, in each direction, of the figure just constructed, interchanging, however, the shaded and non-shaded regions in such a manner that, in the complete figure, of any two adjacent triangles, one shall be shaded, the other non-shaded. Notice that, from the manner in which the figure was constructed, there can be no overlapping, not merely from step to step (*im Kleinen*), but after any series of steps (*im Grossen*).

It remains to prove that the two complete figures are identical. Each pair of adjacent triangles in the second figure are symmetric, either by construction or because they are the images of a symmetric pair. If then in the generation of the old figure we make the same shaded triangle (1) of the new figure the point of departure, as much of the old figure as at any given step has been generated will coincide throughout with a part of the new figure, and hence it can readily be shown * that the old figure coincides throughout with the new one.

If now a fixed point $\omega^{(0)}$ of the linear transformation lay in the upper half-plane, we could connect $\omega^{(0)}$ with ω_0 by a path lying in that half-plane, mark the corresponding path $\varkappa$ joining $x^{(0)}$ with x_0 on the Riemann's surface, add to this the path λ above considered, and then add to λ a path $\varkappa'$ congruent to $\varkappa$, but lying, of course, in a different leaf. Let ω, regarded as a function of x, be continued from $x^{(0)}$ along the complete path $\varkappa\lambda\varkappa'$. Then ω, starting at $x^{(0)}$ with the value $\omega^{(0)}$, will end at $x^{(0)}$ with the value $\omega'^{(0)} = \omega^{(0)}$. Hence the corresponding points of the x-surface (*i. e.*, the extremities of the $\varkappa\lambda\varkappa'$-path) must lie in the same leaf, and here is a contradiction.

We are now ready to form the function $\varphi(\omega)$. Begin with the case

$$\omega' = \omega + k.$$

Then it is sufficient to set

$$\varphi(\omega) = e^{\frac{2\pi i}{|k|}\omega}.$$

Since the pure imaginary part of ω is always positive,

$$R\left(\frac{2\pi i}{|k|}\omega\right) < 0,$$

* The point is that the limiting figure approached in the generation of the old figure cannot be a *part*, but must be the *whole* of the new figure.

where R denotes the real part of the complex quantity. Hence

$$|\varphi(\omega)| = e^{R\left(\frac{2\pi i}{|k|}\omega\right)} < 1.$$

Similarly, if

$$\frac{1}{\omega' - \omega^{(0)}} = \frac{1}{\omega - \omega^{(0)}} + k,$$

we may set

$$\varphi(\omega) = e^{\frac{2\pi i}{|k|}\frac{1}{\omega - \omega^{(0)}}}$$

For the hyperbolic transformation $\varphi(\omega)$ may be defined as follows:

$$\varphi(\omega) = e^{\frac{2\pi i}{|\log k|}\log\frac{\omega - \omega^{(1)}}{\omega - \omega^{(0)}}}$$

This function, to be sure, is not a single valued function of ω; but it is a single valued function of z. For, ω remaining always in the positive half-plane, the angle of the complex quantity $\frac{\omega - \omega^{(1)}}{\omega - \omega^{(0)}}$ may be taken as lying between 0 and π, provided $\omega^{(0)} < \omega^{(1)}$, and hence the pure imaginary part of $\log\frac{\omega - \omega^{(1)}}{\omega - \omega^{(0)}}$ cannot vary by so much as πi. Thus $|\varphi(\omega)|$ is constrained to lie between two fixed limits. The proof is now complete.

The limitation of the linear transformation $\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta}$ to the above two types could be established in various other ways. Thus it could be shown that, since this transformation carries one triangle over into a second, each triangle is carried over into a different one, and hence no point of the upper half-plane can be a fixed point. Or it could be shown directly that an even number of reflections of the kind here considered yield only parabolic and hyperbolic transformations of the above kind. But the complete discussion of the automorphic figure is valuable for other purposes. One property of this figure which can now be easily established has not explicitly come into play in the foregoing, namely, that a reflection on any side of any triangle carries the figure over into itself, except that shaded and non-shaded regions are interchanged.

It is hardly necessary to add that the methods set forth in this lecture are to a large extent those which Klein has done so much towards making a vital force in the modern theory

of functions which Riemann initiated. Cf. Klein-Fricke, Modulfunctionen, vol. 1.

Another noted theorem of Picard's is the following:* *If between two single valued analytic functions of a single variable there exists an algebraic relation of deficiency greater than unity, these functions cannot have an isolated essentially singular point.* Picard published a proof in Darboux' *Bulletin* (1883), which however is incomplete in an essential respect. He gave two other proofs later in the *Acta* (l. c.), the first of which is based on properties of the automorphic functions. It would be interesting to apply the methods of this lecture to the proof of that theorem also.

Lectures II and III.

The Representation of Multiple Valued Functions by Means of Single Valued Functions of a Parameter, treated Geometrically by Riemann's Methods. Poincaré's Theorem.

POINCARÉ'S THEOREM.† *Let $y = f(x)$ be any analytic function of x whatsoever. Then it is possible to find two single valued analytic functions of a parameter z such that, if*

$$x = \varphi(z),$$
$$y = \psi(z),$$

each value of z pertaining to the domain of definition T of the function $\varphi(z)$ will yield a point (x, y)‡ of the analytic configuration $y = f(x)$ and, conversely, to each point of this analytic configuration will correspond, in general, one or more values of z pertaining to T, two such values of z never becoming coincident. There will be at most three values of x at which some branches of $f(x)$ are analytic and such that any point (x, y) of the configuration in which one of these participates will fail of such representation. The domain T will be the interior of the unit circle, or a domain included within that circle.

If, in particular, there are at least three values of x which are singular points for all branches of the function, then the analytic configuration $y = f(x)$ is represented adequately by the functions $\varphi(z)$, $\psi(z)$, in that each point (x, y) of the configuration has cor-

* Picard, "Démonstration d'un théorème général sur les fonctions uniformes liées par une rélation algébrique"; *Acta Math.*, vol. 11 (1888).

† Poincaré: "Sur un théorème de la théorie générale des fonctions;" *Bulletin de la Soc. math de France*, vol. 11 (1883).

‡ This term is intended to express a generalization of the familiar idea of a point of an algebraic configuration (*Gebilde*), and means a value of x with which is coupled a value of the function $y = f(x)$ that belongs to a branch of that function which is analytic at x.

responding to it at least one point z of T, two such values of z never becoming coincident. But even then the neighborhood of a branch point of finite order in the x-surface does not go over into the neighborhood of a point in T.

Before entering on the discussion of this theorem, let us consider some examples of representation by means of a parameter.

1. Suppose $f(x)$ is multiple valued and has two singular points. These may be thrown to 0 and ∞ ; and now two cases arise :

(a) $f(x)$ has but a finite number of values, m ;

(b) $f(x)$ is infinitely multiple valued.

In the first case, the representation is effected by means of the equations

$$x = z^m,$$
$$y = f(z^m) = \psi(z) ;$$

in the second case, by the equations

$$x = e^z,$$
$$y = f(e^z) = \psi(z).$$

2. A further example is furnished by the unicursal curves

$$x = \varphi(z),$$
$$y = \psi(z),$$

where $\varphi(z)$, $\psi(z)$ are rational functions of z, and $z = R(x, y)$.

In each of the above cases, to each value z (T consisting of the whole plane with the exception of a finite number of points) corresponds in general a point (x, y) of the analytic configuration, y being in this point an analytic function of x ; and, conversely, to each such point (x, y) corresponds in general one, and never more than one, value of z.

3. Next may be mentioned the representation of the coordinates of algebraic curves by the elliptic functions when $p = 1$, and, generally, by automorphic* functions. Here the relation between (x, y) and z continues to be one-to-one *im Kleinen*, but is one-to-infinity *im Grossen*.

Turning now to Poincaré's investigations, we find that Poincaré begins by constructing a Riemann's surface for the function $f(x)$ in a novel manner. Let x_0 be a point at which $f(x)$ is analytic ; let a, b, c be any three values of x, intro-

*Cf., for example, Klein, Ueber lineare Differentialgleichungen der zweiten Ordnung 1894, (lithographed); in particular, p. 515, where Picard's "Methode des Linienelements" is set forth.

duced for a purpose to be explained later, and let them be regarded as singular points of each leaf. Consider a path that starts from x_0 and returns to x_0. Such paths may be divided into two classes:

(a) paths that can be contracted to the point x_0 without sweeping over a singular point of the function or one of the points a, b, c;

(b) all other paths.

In the case of paths of the first class only shall the terminal point be said to belong to the same sheet of the Riemann's surface as the initial point.

The surface thus constructed will be infinitely many leaved; it will be bounded by no other isolated singular points than branch points of infinitely high order, these points not being regarded, of course, as belonging to the surface; and thus the function $f(x)$ will be analytic at each point of the surface. Any closed curve on the surface can be drawn together continuously to a point, and thus the function $\int_{x_0}^{x} f(x)dx$ will be single valued on this surface.

Next, a set of curves containing x_0 is constructed bounding regions C_1, C_2, $\cdots$ such that each region is contained in each of its successors and each point of the Riemann's surface just constructed is ultimately contained in a C_n. That such a set of contours can always be constructed will be shown presently.

Poincaré then constructs for each C_n the Green's function u_n with its singularity in x_0 and shows that, E being any region of the surface, u_n converges uniformly on E

$$\lim_{n=\infty} u_n = u.$$

His analysis at this point is allied with that by which Harnack proved his noted theorem* that a function u_n which is harmonic throughout a region for all values of n, which always increases when n increases, and which at one point of the region converges toward a limit, converges at all points within the region, and the limiting function is also harmonic. In fact, in the light of Harnack's theorem it is sufficient for Poincaré to show that the constantly increasing harmonic function u_n converges toward a limit for one single point of the Riemann's surface,—and this he can do at once by means of his auxiliary function, which is virtually the same function $\omega(x)$ that we constructed in the previous lecture. He

* Cf. Picard, Traité d'Analyse, vol. 2, p. 57.

uses, moreover, only those properties of $\omega(x)$ that were deduced in the first part of that lecture. The three points a, b, c are the singular points of the auxiliary function and were introduced solely for the purpose of proving the convergence of u_n.

Poincaré then shows that the function

$$z = e^{-(u+iv)},$$

where v denotes the conjugate function of u, maps the whole Riemann's surface either on the interior of the unit circle or on a region T contained within it, which does not overlap itself. Thus x appears as a single valued function of z; and y, being single valued on the surface, is likewise a single valued function of z. Notice, however, that while $\varphi(z)$ has a natural boundary, this is not necessarily the case for the function $\psi(z)$.

To show the possibility of constructing the contours, number the rational points of the Riemann's surface.* By a rational point $x = x' + x''i$ is meant one for which x' and x'' are both rational numbers. Begin with those situated within the circle of convergence of the series

$$f(x) = \mathfrak{P}_0(x - x_0).$$

Put them down as the first line of a two-dimensional array (the sort of array used in writing out a doubly infinite series). Next, consider the continuations having these points as centers, and number their rational points (not already numbered), to begin with, in a two-dimensional array, the nth line giving the rational points in the circle of convergence about the rational point numbered n; then contract this array into a single line, the second line of the final array; — etc.† The final array can then be reduced to a single line. And now it is sufficient to take C_n so as to include a circle about the point numbered n whose radius is half the radius of the circle of convergence of the element whose center is that point. For, let P be any point of the Riemann's surface, *i. e.*, a point lying within the circle of convergence of an element $\mathfrak{P}_1(x - x_1)$ which is a continua-

* This is essentially what Poincaré did in numbering the elements (power series) defining the analytic continuation of $f(x)$, whose centers are rational points. Cf. Poincaré: "Sur une propriété des fonctions analytiques," *Rendiconti del circolo matematico di Palermo*, vol. 2 (1888).

† The analysis here employed applies to the analytic continuation of any function whatsoever, and thus fills a gap that frequently remains open in the presentation of the Weierstrassian theory of functions.

tion of the element $\mathfrak{P}_0(x-x_0)$. Draw a circle about P whose radius is one-third that of the radius of convergence of the series $\mathfrak{P}_1(x-x_1)$, and let P' be any rational point situated within this circle, the number of this point being n. Then C_n will include the whole of this circle, and hence the point P.

In the foregoing, the points a, b, c were introduced solely for the purpose of effecting the proof of convergence. Without them, the Riemann's surface, the contours C_n, and the Green's function u_n could have been constructed as before. At a given point of the surface u_n steadily increases as n increases and if it could be shown that in one single point u_n converges toward a limit, when $n=\infty$, we should be in possession of that generalization of Poincaré's theorem which consists in removing the restriction "in general." But it is not by the fact that certain points at which $f(x)$ is analytic (those, namely, for which $x=a, b, c$) or at which $f(x)$ has a simple singularity (a branch point of finite order, for example) have no corresponding point within T that we are constrained to seek the essential limitations of Poincaré's theorem. An ideal theorem would be one by which the multiple leaved domain of definition of the function $f(x)$ is mapped in general conformally on a single leaved domain in the z-plane, the neighborhoods of certain isolated points (including isolated singularities and the point $x=\infty$ if necessary) going over by a well defined geometric law into the neighborhoods of vertices abutting on the boundary of T. Such a theorem exists for the algebraic functions, the auxiliary functions $\varphi(z)$, $\psi(z)$ being automorphic functions with fundamental circle.

Poincaré's proof of the restricted theorem makes no use of other properties of the function $f(x)$ than the situation of the singularities of its various branches, and it thus establishes, in certain cases, the existence of functions belonging to a given Riemann's surface,—a question which for the most general Riemann's surface has not as yet been settled. The simplest case is that in which a simply connected Riemann's surface is given, with no other isolated singularities than branch points of infinite order, and such that at least three values of x are included in no leaf. To such a surface there surely correspond functions.

A further point is the following: Nothing in Poincaré's analysis throws any light on whether the region T fills the whole circle. Suppose that in certain cases it did not. Would there then exist a function $\varphi(z)$ whose domain of definition *is* the unit circle and which has otherwise the

same properties as the old $\varphi(z)$? This question can be answered in the affirmative if the following theorem is true: *The interior of any simply connected region whatsoever can be mapped conformally on the interior of a circle.* The determination of the correctness or the incorrectness of this theorem would be a useful contribution to analysis.

Let us assume that this theorem is true. Then we can give with Klein* an elegant proof of the theorem that the coördinates of an irreducible algebraic curve can be represented as single valued automorphic functions, with fundamental circle, of a parameter z. For, construct Poincaré's surface for such a function $y = f(x)$. If now x describes a path which, though not closed on the surface, nevertheless brings the element $\mathfrak{P}_0(x - x_0)$ back into itself, we thus have a one-to-one and conformal transformation of the whole Riemann's surface into itself. This transformation defines then a similar transformation of the *interior* of the unit circle in the z-plane into itself, and such a transformation can readily be shown to be linear. Hence the function $\varphi(z)$ admits a group of linear transformations into itself and this group has the unit circle as fundamental circle, the generating transformations of the group being finite in number. The function $\varphi(z)$ is then an automorphic function with fundamental circle, and the same is true of $y = \psi(z)$.

Klein's reasoning is, however, of much wider scope than is here indicated. It applies to any function such that, if it is continued analytically along a path from x_0 back to x_0, the element $\mathfrak{P}_0(x - x_0)$ comes back to its original value without the path's closing on Poincaré's surface.

The lecture closed with the suggestion of a line of thought which had for its object a possible test of the correctness of the first generalization of Poincaré's theorem above proposed.

Lectures IV and V.

On some Recent Study of the Relation between the Properties of a Function defined by a Power Series and the Coefficients of that Series.

Let the constants a_0, a_1, a_2, etc., be given, and let a function be defined by the series

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots.$$

The first question to be considered is of course that of con-

* l. c. p. 519 et seq.

vergence. Cauchy* in 1821 established a theorem, which can be stated best in terms introduced by G. Cantor. Represent in the usual manner by a set of points on a line the values $|a_n|^{\frac{1}{n}}$, $(n = 1, 2, 3, \cdots)$, and let the largest value belonging to a point of condensation† of the set be denoted by A; then if $A > 0$, the series will converge when $|x| < \frac{1}{A}$ and diverge when $|x| > \frac{1}{A}$, while if $A = 0$, the series will converge for all values of x. This gives as a necessary and sufficient condition for the convergence of the series for all values of x, that $\lim_{n=\infty} |a_n|^{\frac{1}{n}} = 0$.

The foregoing theorem was obtained recently *de novo* by Hadamard‡. He applies it to the further question: has the function defined by the series a singularity at a specified point upon the circle of convergence? The necessary and sufficient conditions for such singularity being deduced, it is possible at pleasure to produce series defining functions having singular points everywhere dense on their circle of convergence, and hence having this circle as a natural boundary. Such a function is given by the series $\Sigma a_\mu x^{\mu}$ if the further condition is satisfied:

$$\frac{c_{\mu+1} - c_\mu}{c_\mu} > \varepsilon > 0$$

for all § values of μ greater than some fixed value $\bar{\mu}$; that is, if with increasing μ the ratio $c_{\mu+1} : c_\mu$ becomes and remains greater than some quantity itself greater than unity.

A more general sufficient condition has been obtained by Fabry‖, namely, the condition that $c_{\mu+1} - c_\mu$ shall increase without limit as μ increases. This and the former constitute sufficient conditions; necessary conditions also have been obtained, but only of a very general type.

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* It is interesting to note that Cauchy here had the notion of a Cantor's set and of a derivative which might consist of more than one point.

† Analyse algébrique (Paris, 1821), pp. 59, 143, 151. Cf. also Résumés analytiques (Turin, 1833), p. 47.

‡ "Essai sur l'étude des fonctions données par leur développement de Taylor," thesis in *Liouville's Journal*, 4th series, vol. 8 (1892).

§ This condition, Hadamard points out, can be stated in somewhat more general form.

‖ "Sur les points singuliers d'une fonction donnée par son développement et l'impossibilité du prolongement analytique dans des cas très généraux." *Annales de l'École normale supérieure*, 1896.

Returning from this application, I wish next to call attention to the study of the simplest singularities of the function on the circle of convergence, namely poles. Darboux had given in 1878* necessary conditions that the function represented by the series should have on the circle of convergence one pole of specified order, but no other singularity; and he had further considered the case of p poles, and also of a branch point of finite order. The question of the sufficiency of his conditions Darboux did not raise, but Hadamard establishes it (l. c. § 14) for the case of a single pole of the first order. A simple pole occurs namely on the circle of convergence of the series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

when and only when two conditions are satisfied:

(a) $\frac{a_\mu}{a_{\mu+1}}$ approaches a finite limit, say x_1, for $\mu = \infty$;

(b) $$\left| \frac{a_\mu}{a_{\mu-1}} - \frac{1}{x_1} \right|^{\frac{1}{\mu}}$$

becomes and remains, as μ increases, less than a quantity inferior to unity.

More complicated is his deduction of the precise conditions for the occurrence of poles of aggregate order p and for the absence of other singularities. The notion of a Cantor's set plays here again an important rôle, as also in the problem of finding the zeros of a given power series, a problem which Hadamard solves by aid of the analysis that he has here developed.

In brief, the details of Hadamard's method are as follows. He first establishes a necessary condition that the function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

have poles $x_1, x_2, \cdots$ of aggregate order p on the circle of convergence of the series. Let

$$P_p(x) = \left(1 - \frac{x}{x_1}\right)^{\mu_1} \left(1 - \frac{x}{x_2}\right)^{\mu_2} \cdots$$
$$= 1 + A^{(1)} x + A^{(2)} x^2 + \cdots + A^{(p)} x^p;$$

$$P_p(x) f(x) = \sum_{n=-p}^{\infty} b_n x^{n+p}.$$

* "Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série," *Liouville's Journal*, 3d series, vol. 4 (1878).

Then $$b_m = a_{m+p} + A^{(1)}a_{m+p-1} + \cdots + A^{(p)}a_m,$$
where $$a_{-k} = 0,\ k = 1, 2, \cdots, p.$$

Let ρ, ρ' denote respectively the radii of the circles of convergence of the above series. ρ' is greater than ρ; then, if ϵ denotes an arbitrarily small positive quantity,

$$|a_m| < \left(\frac{1+\epsilon}{\rho}\right)^m \qquad \text{and} \qquad |b_m| < \left(\frac{1+\epsilon}{\rho'}\right)^m$$

for all values of $m > \overline{m}$ (a fixed integer). A function of the coefficients is now formed, defined by the determinant

$$D_{m,p} = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+p} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+p+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m+p} & a_{m+p+1} & \cdots & a_{m+2p} \end{vmatrix}.$$

If the ith column of this determinant, multiplied by $A^{(p+1-i)}$ $(i = 1, 2, \cdots, p)$, is added to the last column, the a's of that column will be replaced by b's of like index. Hence it follows that

$$|D_{m,p}| < \left(\frac{1+\epsilon}{\rho^p\rho'}\right)^m \qquad \text{or} \qquad |D_{m,p}|^{\frac{1}{m}} < \frac{1+\epsilon}{\rho^p\rho'},$$

when m exceeds a certain fixed value, and thus a necessary condition is obtained, which can be formulated as follows:

Represent by a set of points the values $|D_{m,p}|^{\frac{1}{m}}$; *then the largest value belonging to a point of condensation of the set is less than* $1/\rho^{p+1}$ *and does not exceed* $1/\rho^p\rho'$.

Conversely, consider the Cantor's set

$$|D_{m,P}|^{\frac{1}{m}}, \qquad (m = 1, 2, \cdots)$$

for each value of P: $1, 2, \cdots$. The largest value λ belonging to a point of condensation is surely not greater than $1/\rho^{P+1}$. It may for some values of P be less. Let p be the *smallest* such value. *Then* $f(x)$ *has no other singularities than poles on the circle of convergence, and their aggregate order is* p. *With the exception of these poles,* $f(x)$ *is analytic throughout a circle of radius* ρ', *where* $\lambda = 1/\rho^p\rho'$.

The proof of this theorem rests on the following reasoning. First the lemma is established that $|D_{m,p-1}|^{\frac{1}{m}}$, $m = \infty$, actually converges toward $1/\rho^p$. Hence the p equations which Hadamard introduces at this point

$$a_{m+p+i}+A_m^{(1)}a_{m+p-1+i}+A_m^{(2)}a_{m+p-2+i}+\cdots+A_m^{(p)}a_{m+i}=0$$

will determine p quantities $A_m^{(1)}$, $A_m^{(2)}$, $\cdots$, $A_m^{(p)}$ as functions of m. These quantities are shown to converge respectively towards limits $A^{(1)}$, $\cdots A^{(p)}$, when $m=\infty$, and the quantity

$$b_m = a_{m+p}+A^{(1)}a_{m+p-1}+\cdots+A^{(p)}a_m$$

is such that $$|b_m|<\left(\frac{1+\varepsilon}{\rho'}\right)^m, \qquad (\rho'>\rho.)$$

Thus if the polynomial

$$P_p(x)=1+A^{(1)}x+A^{(2)}x^2+\cdots A^{(p)}x^p$$

is formed and multiplied into $f(x)$, the product

$$P_p(x)\,f(x)=\sum_{m=-p}^{\infty}b_m x^{m+p}$$

will have a circle of convergence of radius $\rho'>\rho$. This establishes the proposition.

Hadamard then proceeds to obtain the necessary and sufficient condition that $f(x)$ should have on the second circle, the circle of convergence of the b-series, only poles. He makes this determination directly, not by a mere repetition of the previous reasoning. And so on. The discussion of the results is easy. Perhaps one of the most striking consequences is *the establishment of the necessary and sufficient condition that the function $f(x)$, defined by the power series $\sum_{m=0}^{\infty}a_m x^m$ and its continuations, should have no other singularities in the finite region of the plane than poles, and the determination of the position and order of each of these poles.*

As an application of the foregoing Hadamard determines the zeros of a function, defined by a power series

$$\psi(x)=C_0+C_1x+C_2x^2+\cdots,$$

that lie within the circle of convergence of that series, explicitly in terms of the C's.

Let $$f(x)=\frac{1}{\psi(x)}=a_0+a_1x+a_2x^2+\cdots$$

Here a_m can be expressed rationally in terms of C_0, C_1, ..., C_m. Next $D_{m,p}$ is expressed rationally in terms of the C's and thus a solution of the problem, of a nature similar to that

of the solution of the main problem above discussed, is obtained.

Daniel Bernoulli had pointed out that if one root x_1 of an algebraic equation is in absolute value smaller than any other root, it can be obtained as the limit when $m = \infty$ of a variable which is a rational function, for each value of m, of the coefficients of the equation, namely:

$$x_1 = \lim_{m=\infty} \frac{s_{-m}}{s_{-m-1}},$$

where $$s_i = x_1^i + x_2^i + \cdots + x_n^i.$$

Runge* generalized Bernoulli's results. Hadamard's results amount to a still further generalization to power series.

It is worthy of note that the criteria occurring in the above theorems of Hadamard's differ from those usual in the lower analysis in that they do not turn upon the existence and character of *a single limiting point*, but rather upon those of *a set of such points, namely the cluster points of a Cantor's set.*

Two theorems of Hadamard's† next to be considered concern integral functions, whether rational or transcendental. As before, denote by $F(x)$ a function explicitly defined by the series

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m + \cdots,$$

and let the series converge for all finite values of x. The necessary and sufficient condition for this was seen to be that

$$\lim_{m=\infty} |a_m|^{\frac{1}{m}} = 0.$$

Of all points x not lying without a circle of radius R about the origin, there is at least one—call it x—upon the circumference, in which $|F(x)|$ attains the maximum value. That maximum value increases indefinitely with the radius R; but how rapidly? As to the manner of its increase Hadamard establishes the theorem: *The absolute value of $F(x)$ for points x, $|x| = R$, becomes infinite less strongly than the expression*

* *Acta Math.*, vol. 6 (1885).

† "Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann," *Liouville's Journal*, 4th series, vol. 9 (1893), p. 171 seq.

$$R^{\epsilon} \cdot e^{\int_{R_0}^{R} \frac{\psi(R)}{R}\, dR}$$

where ϵ is any arbitrarily small positive constant, and $\psi(R)$ is an increasing positive function of R depending upon the coefficients $a_0, a_1, a_2, \cdots$ in the following manner. It is always possible to construct a positive continuous function* $\chi(\xi)$ such that

(a) if $a_m \neq 0$, $\qquad \chi(m) \leqq \dfrac{1}{|a_m|}, \qquad (m \geqq \overline{m})$;

(b) the function

$$\eta = \varphi(\xi) = [\chi(\xi)]^{\frac{1}{\xi}}$$

steadily increases as ξ increases, and becomes infinite with ξ. The inverse of the function φ,

$$\xi = \Psi(\eta),$$

will then, for values of η that exceed some fixed inferior limit R_0, be a single valued function of η, increasing indefinitely with η; and this is the function Ψ whose definition completes the statement of the above theorem. In place of the particular Ψ so defined we may substitute any increasing function which becomes infinite more strongly, and the theorem will hold true *a fortiori*.

An example is this: if from and after the value $m = \overline{m}$ the condition is satisfied

$$|a_m| \leqq \frac{1}{(m!)^{\alpha}},$$

where α is some positive constant, then $|F(x)|$ becomes infinite less rapidly than $e^{R^{\beta}}$, R denoting $|x|$ and β being any constant greater than $\dfrac{1}{\alpha}$.

A second theorem given by Hadamard is based upon the lemma: if the real part of an integral function $G(x)$ is algebraically less than R^{λ} ($R = |x|$) for all values of x whose R exceeds some fixed inferior limit, then $G(x)$ is a polynomial of degree not higher than λ. The theorem itself is as follows. *If $F(x) = e^{G(x)}$, where $G(x)$ denotes an integral function, and if for all values of x having $|x| = R > \overline{R}$ the absolute value of $F(x)$ is less than $e^{R^{\lambda}}$, then $G(x)$ is a polynomial of de-*

* Hadamard's analysis at this part of his memoir requires revision. The results appear to be correct. The account that I gave in the lecture of the construction of the function $\chi(\xi)$ must be essentially modified in details, but the general method was, I believe, correct.

gree not higher than λ. It may be remarked in passing that for such integral functions $F(x)$ Picard's theorem is at once seen to be true, since the equation

$$e^{G(x)} = c + 0,$$

where c is arbitrary and G is a polynomial, always has roots.

The third theorem to which we now turn is the great theorem of this paper. It relates to the zeros of an integral function. Denote by ρ the absolute value of any zero x_p of the integral function

$$F(x) = \sum_{m=0}^{\infty} a_m x^m;$$

and arrange the zeros in a definite order, giving to each the corresponding index p in such a manner that always $\rho_{p+1} \geqq \rho_p$. *If a function* $\varphi(m)$, *monotonic when* $m > \overline{m}$, *be so chosen that*

$$\frac{1}{\varphi(m)} \geqq |a_m|^{\frac{1}{m}}, \qquad (m > \overline{m}),$$

then, for any arbitrarily small positive quantity ϵ, *an inferior limit* $\overline{p}$ *can be found such that, for* $p > \overline{p}$,

$$\rho_p > (1 - \epsilon) \cdot \varphi(p).$$

An illustration of this third theorem is given in any series having

$$|a_m| \leqq \frac{1}{(m!)^\alpha}.$$

Assuming any α' such that

$$0 < \alpha' < \alpha,$$

and taking

$$\varphi(m) = m^{\alpha'},$$

there results $\rho_p > p^{\alpha'}$, and the *genre** of $F(x)$ is an integer not exceeding $\frac{1}{\alpha}$.

The impulse to the inquiries culminating in these theorems was given by two theorems of Poincaré,† inspired by the researches of Weierstrass ‡:

*A term introduced by Laguerre, *Comptes Rendus*, vol. 94 (1882), p. 160. Cf. Forsyth's Theory of Functions, § 59 et seq.

† "Sur les fonctions entières :" *Bulletin de la Soc. math. de France*, vol. 11 (1883)

‡ "Zur Theorie der eindeutigen analytischen Functionen;" *Abh. d. Berliner Akad.*, 1876; Math. Werke, vol. 2, p. 7.

(*a*) If $F(x)$ has the *genre* N, then

$$|F(x)| < k \cdot e^{\epsilon R^{N+1}}$$

for all values of x having $|x| = R > \overline{R}$, where k, ϵ denote any real positive constants.

(*b*) If $F(x) = \sum_{m=0}^{\infty} a_m x^m$ has the *genre* N, then

$$|a_m| < \frac{\epsilon}{(m!)^{\frac{1}{N+1}}}$$

where ϵ is taken arbitrarily small, provided m is sufficiently large.

Poincaré also raised certain questions regarding the *genre* of sums and derivatives of functions whose *genre* is given. Concerning Poincaré's two theorems Hadamard shows that with slight modifications the converse theorems are true.

Borel has studied a number of questions akin to those that we have here considered, and has carried further some of the investigations instituted by Hadamard. I refer to his memoir in the *Acta Mathematica*, vol. 20 (1897): "Sur les zéros des fonctions entières."

Hadamard closes his memoir with a valuable application of the results he has obtained to the study of Riemann's function $\zeta(s)$.

Lecture VI.

On certain Cantor's Sets, and their Application in a Question concerning Cauchy's Definition of an Analytic Function.

In investigations in the general theory of functions, either of real or of complex variables, a clear insight into the possibilities that Cantor's sets present is indispensable. It is here less a question of systematic theoretical developments than of a body of knowledge of facts given by concrete examples. I should like to present to you today one or two examples that I have frequently found useful in work of a general nature.*

* The last of these examples is, for instance, useful in forming a critique of the classification of Cantor's sets which Painlevé adopts (Cf. his Thesis: "Sur les lignes singulières des fonctions analytiques," Paris, 1887; *Toulouse Annales*, vol. 2 (1888)). The division into the three classes, *ponctuel, linéaire, superficiel* suggests that a set which is *ponctuel*, for example, is *a fortiori, linéaire*, and *for that reason* is excluded from that class. That would surely not be the case if Painlevé had used the square instead of the circle in defining *ponctuel*, as the last of my examples shows. If on the other hand, the set of that example is in fact not *ponctuel*, then it fails to fall under any one of the three classes.

I will begin with a set of points on a line, the conception of which enabled me to give the first example,* so far as I know, of a series of real terms

$$u_1(x) + u_2(x) + \cdots$$

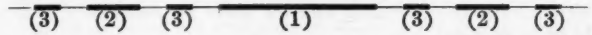
of the following sort: for each value of x in the interval $a \leqq x \leqq b$ each term of the series is continuous and the series converges toward a continuous function $f(x)$; the term-by-term integral

$$\int_{x_0}^{x} u_1(x)dx + \int_{x_0}^{x} u_2(x)dx + \cdots$$

likewise converges toward a continuous function, $\varphi(x)$; but

$$\varphi(x) + \int_{x_0}^{x} f(x)dx.$$

This set is a special case of a set given by Harnack† and is constructed as follows. It shall lie in an interval L of unit length.



First Step. In the middle of this interval lay off an interval (1) of length

$$l_1 = \lambda - \tfrac{1}{3}\lambda,$$

where λ is chosen arbitrarily as a positive quantity not greater than the unity: $0 < \lambda \leqq 1$.

In particular, let $\lambda = \frac{3}{4}$; $l_1 = \frac{1}{2}$.

Second Step. In the middle of each of the free end-intervals lay off an interval (2), both of these intervals to be of the same length l_2 and such that the total length of the intervals (1), (2) is

$$l_1 + 2l_2 = \lambda - \tfrac{1}{4}\lambda.$$

n-th Step. In the middle of each of the equal free intervals lay off an interval (n), all of these intervals to be of the same length l_n and such that the total length of the intervals (1), (2), $\cdots$, (n) is

$$l_1 + 2l_2 + 2^2l_3 + \cdots + 2^{n-1}l_n = \lambda - \frac{1}{n+2}\lambda.$$

* Cf. § 20 of my paper: "Non-uniform convergence and the integration of series term by term," *Amer. Jour. of Math.*, vol. 19 (1897).

† *Math. Ann.*, vol. 19, p. 239.

When n increases indefinitely, a set of intervals is obtained and their extremities form a Cantor's set. This set is enumerable. It is not *perfect*, *i. e.*, does not coincide with its derivative. Let all the limiting points not already included in the set be annexed. Then we have a perfect set, and *this set is the first example I wish to show you.*

The content of this set is* $1 - \lambda$ *and hence is not, in general,* 0.

In order to obtain an example of a two-dimensional set which is nowhere dense in two dimensions, *i. e*, is such that in every neighborhood of any point of the set there exist two-dimensional regions which are free from points of the set, it is sufficient merely to erect at each point of the above set a perpendicular of unit length to the line on which the set lies, all the perpendiculars being drawn on the same side of this line. But it not infrequently happens that the simplest example is not the most striking one. And so here. You would prefer to see a set of points containing no lines. Very well. Take the perpendicular at one extremity of L as a new interval L' and construct in L' a set of points like the set just constructed in L. Through each point of this set draw a parallel to L; *the points of intersection of these parallels with the above perpendiculars form the set in question.*

It is readily shown that *the content of this set is* $(1 - \lambda)^2$, and is then in general not 0. Moreover, if ϵ is an arbitrarily small positive quantity and P any point of this set, a rectangle can obviously be drawn containing P, but having no points of the set on its boundary, and having its sides less than ϵ in length.

As an instance of the use of such conceptions of Cantor's sets as those just considered may be mentioned a question that arises in the Cauchy definition of an analytic function. Let $f(z)$ be a single valued function (the word *function* being used here in its most general sense), defined at each point of a two-dimensional region T, and let it have, at each point of T, a finite derivative, *i. e.*, for any such point z, let

$$\frac{f(z+h) - f(z)}{h}$$

* *Amer. Jour.*, l. c. § 17. Let the interval L be divided into $1/2^m$ equal parts and denote the sum of those parts to which points of the set pertain by s_m. Then the *content* of the set is defined as $\lim_{m=\infty} s_m$. The extension of the definition to two-dimensional sets is at once obvious, the above subintervals being replaced by squares $1/2^m$ on a side; etc.

converge toward a finite limit $f'(z)$ when h approaches 0. What next? It is customary to add the condition that $f'(z)$ *be continuous in* T; for without this condition no proof of Cauchy's integral theorem has as yet been given. Now the question is whether this further condition is not superfluous; *i. e.*, whether the continuity of $f'(z)$ does not follow from the preceding conditions, or at least from those coupled with a less comprehensive condition; as for example, that $f'(z)$ remain finite in T

$$|f'(z)| < G,$$

where G is some fixed quantity. That a little less than the assumption of exceptionless continuity will suffice is easily seen. For, from the existence of the derivative follows the continuity of the function $f(z)$. Now it is both a necessary and a sufficient condition for the continuity of $f'(z)$ that $\int f(z)dz$, taken along any closed path in T that can be drawn together continuously to a point of T, should vanish.* Hence it is readily shown that if $f'(z)$ is in general continuous in T, but behaves in an unknown manner at isolated points or along curves that are not too crinkly, then $f'(z)$ will be continuous everywhere in T. We can go a step further and state the theorem: *If* $f'(z)$ *is in general continuous in* T, *the points about which no supposition is made forming a set of content null, and if*

$$|f'(z)| < G,$$

where G *is a fixed quantity, then* $f'(z)$ *is everywhere continuous in* T.† For Goursat's proof of Cauchy's integral theorem applies, with a slight modification, to this case.

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* Cf. "Some points in the elements of the theory of functions," this BULLETIN, 2d Ser., vol. 2, 1895-96. At that time a paper by G. Morera "Un teorema fondamentale nella teorica delle funzioni di una variable complessa," *Real. Ist. Lomb. Rend.* (2) vol. 19, Apr., 1886, was unknown to me, in which the author not only establishes the above condition as sufficient, but applies his results to prove Weierstrass's theorem that a series of functions, analytic throughout a two-dimensional region, which converges uniformly within this region, defines a function analytic within the region.

† Professor Chessin states this theorem without the restriction that $|f'(z)| < G$. (Cf. *Annals of Math.*, vol. 11, 1896.) His proof, which is given by a familiar method, is based on the following theorem (l. c., pp. 52, 53), which he ascribes to Harnack, but which Harnack neither states nor proves: *Let* $X(x, y)$ *and* $Y(x, y)$ *be two functions of the real variables* x *and* y, *which are finite and continuous throughout a connected domain* (D), *and which generally* (*i. e.*, with the exception of points and lines forming a discrete multiplicity) *satisfy the equation* $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$; *let further* (x_0, y_0)

Finally, we may add the following theorem, which includes the preceding one as a special case: *If $f'(z)$ exists at each point of T and if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ (where $f(z) = u + vi$) are each capable of surface integration over any two dimensional region contained in T, as well as of partial integration with regard to x and y (i. e., line integration parallel to either axis), then $f(z)$ is analytic in T.*

For u and v are continuous functions of x and y in T, satisfying the relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

each of the integrals

$$\int (udx - vdy), \quad \int (vdx + udy),$$

taken along a closed path that cuts out from T a simply connected region, can be shown by Green's theorem* to vanish; and hence $\int f(z)dz$, taken along such a path, vanishes too.

This theorem is in one respect more general than any of the preceding in that it makes no supposition about the continuity of $f'(z)$ whatsoever. It meets its limitations, however, already in the case that, $f'(z)$ being in general continuous in T, nothing is known about the behavior of $f'(z)$ in the points of a Cantor's set of positive content, —for example, the above set when $\lambda < 1$.

It may be remarked that the existence of the derivative $f'(z)$ involves more than the mere existence of the partial derivatives of u and v, and of the relations between them.

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and (x, y) be any two points within (D); then the definite integral

$$\int_{(x_0, y_0)}^{(x, y)} (Xdx + Ydy)$$

will be independent of the path of integration, provided the several paths lie entirely within (D) and can be brought to coincide with one another by a continuous deformation without crossing any of the boundary lines of (D). Without introducing further restrictions there is at present no prospect of obtaining a proof of this theorem (if indeed it be true).

*The proof of Green's theorem in the generality here required depends on the following theorem: If $\phi(x)$ is defined for each value of x in the interval: $a \leqq x \leqq b$, and if $\phi(x)$ has in each of these points a finite derivative $\phi'(x)$; if furthermore $\phi'(x)$ is integrable in the interval, then

$$\int_{x_0} \phi'(x)dx = \phi(x) - \phi(x_0).$$

It involves the existence of a complete differential of the functions u and v. Cf. Stolz, Differential und Integralrechnung, vol. 1, ch. 4, § 8 and vol. 2, ch. 12, § 7.

To test the truth of a general theorem one mode of procedure is to form an example which proves the theorem to be false. From the foregoing it appears that in the present case simpler examples are surely impossible than those

(a) in which $f'(z)$ becomes discontinuous along a curve whose tangent does not turn continuously along any arc;*

(b) in which, $f'(z)$ being assumed to remain finite in T, $f'(z)$ becomes discontinuous in a Cantor's set of positive content;

(c) in which not both of the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are capable of surface integration over any region lying in T, or of line integration along a line parallel to one of the coördinate axes.

HARVARD UNIVERSITY,
September, 1898.

THE FIFTIETH ANNIVERSARY MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE semi-centennial meeting of the American Association for the Advancement of Science was held in Boston, August 22–27. There were over 900 members and associates registered. A large number of members of affiliated societies were also in attendance, many of whom did not register, but had the privilege of taking part in the proceedings of the sections in which they were interested. Their presence increased the general scientific interest; and it was the evident desire of the various sections to have still closer relations with the respective affiliated societies, as it is felt that any tendency towards the isolation of groups of specialists may partially defeat one of the objects the Association has at heart, viz., the spread of a popular interest in the work of scientific men.

The section of mathematics and astronomy, and its near neighbor, the physical section, were well attended, the

* It is possible that by analysis similar to Jordan's, Cours d'Analyse, vol. 1, §§ 193–196, this condition may be made more general.

programmes being fairly representative of the various phases of American thought and activity in the mathematical and physical sciences.

The officers of Section A were: Vice-President, E. E. Barnard; Secretary, Winslow Upton; Councillor, E. C. Pickering; Sectional committee, E. E. Barnard, W. Upton, J. McMahon, C. L. Doolittle, H. M. Paul, G. E. Hale; Committee to nominate officers of section, E. E. Barnard, W. Upton, G. C. Comstock, J. R. Eastman, R. S. Woodward.

The opening address of the Vice-President was on "The development of astronomical photography."

Of the various contributions accepted by the committee twelve are in pure mathematics, thirteen in applied mathematics, and sixteen in astronomy. A few of the titles that appeared on the preliminary programme are not reproduced here on account of the absence both of the authors and of the finished papers. For convenience of grouping, the numbers are rearranged as follows:

(1) "Graphical logic." By Professor ELLEN HAYES, Wellesley College, Wellesley, Mass.

(2) "On the operation groups of order 48 and those of order $2p^3$, p being any prime number." By Dr. G. A. MILLER, Cornell University, Ithaca, N. Y.

(3) "Illustrations of the comitant method of constructing the imaginary loci of analytical geometry, so as to render their properties evident to the eye." By Professor FRANK H. LOUD, Colorado College, Colorado Springs.

(4) "On the aims of the international society for the promotion of quaternions and allied branches." By Dr. ALEXANDER MACFARLANE, Lehigh University, South Bethlehem, Pa.

(5) "Linear transformations in four dimensions." By Professor ARTHUR S. HATHAWAY, Rose Polytechnic Institute, Terre Haute, Ind.

(6) "On rational right-angled triangles, II." By Dr. ARTEMAS MARTIN, U. S. Coast Survey, Washington, D. C.

(7) "Some notes on 'direction.'" By S. EDWARD WARREN, C.E., Newton, Mass.

(8) "The general theory of anharmonics." By Professor EDGAR ODELL LOVETT, Princeton University, Princeton, N. J.

(9) "A ternary and a quaternary linear group simply isomorphic to the group of linear fractional substitutions in the Galois field of order p^n." By Dr. LEONARD EUGENE DICKSON, University of California, Berkeley, Cal.

(10) "On harmonic functions." By Dr. ROLLIN A. HARRIS, U. S. Coast and Geodetic Survey.

(11) "A proposed tidal analyzer." By Dr. ROLLIN A. HARRIS.

(12) "A tidal abacus." By Dr. ROLLIN A. HARRIS.

(13) "The harmonic analysis of high and low waters." By Dr. ROLLIN A. HARRIS.

(14) "The mass and moments of inertia of the earth's atmosphere." By Professor R. S. WOODWARD, Columbia University, New York City.

(15) "Two new forms of apparatus for measuring the acceleration of gravity." By Professor R. S. WOODWARD.

(16) "The gravitation constant and the mean density of the earth." By Professor R. S. WOODWARD.

(17) "The limitations of the present solution of the tidal problem." By JOHN F. HAYFORD, Coast and Geodetic Survey, Washington, D. C.

(18) "Fifty years of American geodesy." By E. D. PRESTON, Coast and Geodetic Survey, Washington, D. C.

(19) "On the duplex base apparatus of the U. S. Coast and Geodetic Survey." By WILLIAM EIMBECK, U. S. Coast and Geodetic Survey, Washington, D. C.

(20) "On the diminution of the refraction of the atmosphere with height and its effect upon trigonometrically determined elevations from reciprocal zenith distances." By WILLIAM EIMBECK.

(21) "Behavior of the atmospheres of gas- and vapor-generating globes in celestial space." By Dr. J. WOODBRIDGE DAVIS, Woodbridge School, New York City.

(22) "Making astronomy popular." By Miss MARY PROCTOR, New York City.

(23) "Correction of local error in stellar photometry." By HENRY M. PARKHURST, Brooklyn, N. Y.

(24) "Variation of latitude at New York City and the constant of aberration from observations with a zenith telescope (Talcott's method) at Columbia University Observatory during the years of 1892–1898." By Professors J. K. REES, HAROLD JACOBY, and Dr. H. S. DAVIS.

(25) "The parallaxes of 61^1 and 61^2 Cygni from a reduction of the Rutherfurd measures." By Dr. H. S. DAVIS, Columbia University Observatory.

(26) "The Præsepe group: measurement and reduction of the Rutherfurd photographs." By Dr. FRANK SCHLESINGER, Columbia University.

(27) "Discordances between the north polar distances of stars derived from direct and reflected observations." By

Professor J. R. EASTMAN, U. S. Naval Observatory, Washington, D. C.

(28) "The treatment of results from reflection observations at the Greenwich observatory." By Professor J. R. EASTMAN.

(29) "A description of the altazimuth instrument recently constructed for the U. S. Naval Observatory." By GEORGE A. HILL, Washington, D. C.

(30) "On a new application of the prismatic camera to total eclipse." By Professor DAVID P. TODD, Amherst College, Amherst, Mass.

(31) "A description of instantaneous azimuth and altitude charts of the heavens." By F. W. COAR, Cambridge, Mass.

(32) "Instruction in elementary astronomy by means of observations made by the student." By W. MAXWELL REED, assistant in astronomy, Harvard College.

(33) "Personal equations during the past century; a brief summary." By Professor TRUMAN H. SAFFORD, Williams College, Williamstown, Mass.

(34) "The condition of the surface of the planet Jupiter." By Professor G. W. HOUGH, Dearborn Observatory. (With lantern slides.)

(35) "The Yerkes Observatory and its work." By Professor GEO. E. HALE, Director of the Yerkes Observatory, Williams Bay, Wis. (With lantern slides.)

(36) "A summary of planetary work at the Lowell Observatory and the conditions under which such work has been performed." By A. E. DOUGLASS, Lowell Observatory, Flagstaff, A. T.

(37) "Astronomy in southern California." By Dr. LEWIS SWIFT, Lowe Observatory, and Echo Mountain, Cal.

(38) "Report on the recent progress in hydrodynamics." By Professor ERNEST W. BROWN, Haverford College, Haverford, Pa.

(39) "Report on theory of invariants: the chief contributions of a decade." By Professor HENRY S. WHITE, Northwestern University, Evanston, Ill.

(40) "Report on the recent progress in the mathematical theory of electricity and magnetism." By Professor ARTHUR G. WEBSTER, Clark University, Worcester, Mass.

(41) "Report on the modern group theory." By Dr. G. A. MILLER. Cornell University, Ithaca, N. Y.

In No. 1 it is pointed out that Mill's fifth "canon of induction" is essentially a mathematical proposition and is

easily presented by means of the Cartesian graph. The ordinary axes of reference must be viewed as cause-effect axes which become specialized for any particular subject of investigation. In the upper right-hand quarter of the field surrounding the origin is a region which may be called the *field of observation*. It represents the investigator's limits of experiment. Exterior to the observation field is the *field of inference*. The curve expressing the concomitant variations of two phenomena—whatever kind of curve it may be—becomes inferential after crossing the boundaries of the field of observation. There is a generic cause-effect curve possessing properties which belong to specific curves, but these properties cannot be incorporated into the abstract equation of pure mathematics $f(x, y) = 0$. The full paper will appear in *Science*.

In No. 2 it is shown that there are only two groups of order 48 that do not contain any subgroup of order 24. Each of these two groups includes 16 conjugate subgroups of order 3 and only one subgroup of order 16. This subgroup is characteristic as well as Abelian. In one of the two groups it contains 15 operators of order 2 and in the other it contains 12 operators of order 4 that do not have a common square. The total number of the groups of order 48 is 52. Since $48 = 3.16$ there are as many Abelian groups of this order as there are of order 16, viz., 5, and each of these groups is the direct product of an Abelian group of order 16 and some operator of order 3. There is one Hamiltonian group of order 48. It is the direct product of the quaternion group and the cyclical group of order 16. The second part of the paper proves that when p is any odd prime number there are just 15 groups of order $2p^3$. Three of them are Abelian, but none is Hamiltonian since the order of a Hamiltonian group must be divisible by 8. Ten of the groups of order $2p^3$ contain Abelian subgroups of order p^3 while only two of them contain cyclical subgroups of this order. Each group of order $2p^3$ contains only one subgroup of order p^3 and five such groups contain only one subgroup of order 2. When $p = 2$ there are only 14 groups of order $2p^3$. These 14 groups have been known for several years. Five of them are Abelian and one is Hamiltonian. These groups now complete the list of all the groups whose orders are less than 64.

No. 3 is a continuation of a paper read at the Madison meeting, 1893, and in the interim an article has appeared

in the *Annals of Mathematics*, vol. 8, No. 2, under the title "A construction for the imaginary points and branches of plane curves," wherein the earlier devices for the representation of imaginary loci are discussed, and the theoretical relationships of the author's method pointed out. The present paper deals with its pedagogical usefulness. Suppose that a beginner in analytic geometry learns first the usual (Argand's) construction for complex quantities, their addition, multiplication, etc., and is then taught to define the point x, y as located by the complex number $x+iy$. This coincides with the Cartesian definition if the coördinates x and y are real, and equally permits the construction of the point if they are imaginary or complex. A linear equation $ax+by+c=0$ is satisfied by one and only one pair of coordinates locating any given point of the plane. Its locus accordingly covers the plane with a single continuous sheet. If $p+iq$ represent the value of x at any point, those points for which $q=0$ all lie in a right line, which is the familiar real locus provided a, b, c be real. All points for which $q=1$ lie on a parallel to this, those for which $q=2$ on a second, etc. These parallels are called "comitants,"—viz: they are the "x-comitant 1," the "x-comitant 2," etc. There is a similar series of y-comitants. If a, b, c are complex, the one series of equidistant parallels is crossed by the other, the two "comitants zero" meeting at the real point. The locus of an equation of second degree is in general two leaved, exemplifying the elementary properties of Riemann's surfaces. The foci are branch points. The comitants are curves of second, third, or fourth order, but points on them are very simply located by direct use of the equation. Diagrams were exhibited illustrating comitants of real and imaginary right lines, and of the three species of conic, with cases of intersection and contact at imaginary points.

In No. 4 Dr. Macfarlane said in part that the province of mathematics which it is proposed to develop and promote is more or less fully covered by such names as quaternions, Ausdehnungslehre, vector-analysis, geometric calculus, directional calculus, space-analysis, etc. These various titles have been used by writers on the subject; what is common to them all is that the branch of analysis treated of is founded on the properties of space. According to the view of the speaker, ordinary algebra is the analysis of the straight line, or anything that can be adequately represented by the straight line; the algebra of the complex quantity is the analysis of the plane, or anything that can be adequately

represented in a plane, that is, analytical plane trigonometry; space analysis is the algebra of quantities in space, or anything that can be adequately represented in space. The elementary part of space analysis consists of analytical spherical trigonometry and is identical with Hamilton's quaternions. The logical connection of quaternions with analytical plane trigonometry was pointed out, the argument being, that, if the latter is algebra, so also is the former. A complementary part of the analysis, undeveloped by Hamilton, deals with analytical hyperboloidal trigonometry. The speaker said that while analytical plane trigonometry corresponds to Euclidean geometry, the spherical and hyperboloidal trigonometries correspond to the two non-Euclidean geometries. From these and other considerations the author contended that the branch of mathematics which it is proposed to develop is not liable to the objection of "being within very narrow bounds," an objection which was given out at a meeting of the German Mathematical Society. On the contrary, space analysis, being the logical generalization of the algebra of the complex quantity, must be of the widest influence in the whole realm of mathematical analysis. The greatest obstacle to progress at present existing is needless diversity of notation; to promote arrangements for arriving at a settled notation is one of the first objects of the society named in the title.

No. 5 applies quaternion analysis to the treatment of continuous transformations that are equivalent to rigid, dilational, and inversional transformations of a fourfold figure. These are shown to be the most general transformations in which corresponding figures are similar in their infinitely small parts. The normal forms, invariant figures, and path curves of these transformations are also determined. With respect to rigid motion in fourfold space it appears that motion about a point has in general no other fixed point. There are two fundamental motions about a point, called a *turn* and *contra-turn*, whose invariant figures are a system of planes through the center, so that one and only one passes through a given central line. The path curves are circles in these planes and all lines are turned through the same angle. Every rigid motion about a fixed point is a combination of a turn and contra-turn. It has two invariant planes, in which every line of one plane is perpendicular to every line of the other. The angle between corresponding lines in one of these planes is the sum, and in the other the difference, of the angles of the

component terms. The path curves of points in these planes are circles, and of points outside these planes spiral curves on circular rings in the boundaries of fourfold spheres that are concentric with the fixed point. When two points are fixed, the angles of the component turns are equal, every point of one invariant plane becomes fixed, and the motion is rotation about this plane as axis. The parallels to the other plane become invariant equators. The path curve of a point is a circle that is concentric with the axis and lies in an equator. The rigid displacement with no point fixed is the resultant of a rotation about a fixed axis and a translation parallel to that axis. It is interesting to note that, as in a plane, symmetric figures as to a center are rigid displacements of each other. Thus a person could be moved rigidly out into fourfold space and back, so as to appear on his return as his image in a mirror appears.

No. 6 proves several properties of right triangles stated in part I of the paper, read last year. It also proves the following two theorems: Every odd number greater than 1 is the shorter leg of a right triangle whose sides are integers prime to each other. The double of any even number except 2 is the shorter leg of a right triangle whose sides are integers prime to each other. The following five problems are fully solved: "To find all the right triangles whose sides are integers, having one leg equal to 60." (Thirteen such triangles.) "To find any number of rational right triangles having a common leg." [The general expressions for the sides of *two* such triangles are

$$2pqrs, \quad pq(r^2 - s^2), \quad pq(r^2 + s^2);$$
$$2pqrs, \quad rs(p^2 - q^2), \quad rs(p^2 + q^2).]$$

"To find any number of right triangles having the same hypotenuse." "To find a right triangle whose sides and area are integers in arithmetical progression." "To find right triangles whose legs are consecutive integers." [If a, b, c be one solution, the next solution in order of magnitude is $2(a + c) + b$, $2(b + c) + a$, $2(a + b) + 3c$]. The paper will be printed in the *Mathematical Magazine.*

No. 7 argues chiefly against the proposed disuse of the term "direction." The idea of single direction is gained by standing at one point and intently facing another. It is *which way* one point is from another. Turning from facing

one point to facing another we get the idea of difference of direction, or an angle. When by natural movements of work or play, we become familiar with the terms forward and backward, right and left, up and down—each pair representing the pair of opposite senses in which a single line of direction can be taken—we have, in the concrete terms of daily life, the standard guiding directions with which all others are compared. These three constitute a system such that no fourth direction has the same relation to them that they have to each other, viz., the relation that motion on any one of them involves no progress in the direction of the other two. The difference between these directions is the *right angle*, and each is *perpendicular* to the others. *Dimensions* are measures in the independent directions. Moving invariably from one point in the direction of another fixed point, my path has but one direction and is a *straight line*. Sometimes the word "direction" is avoided and a straight line is defined as the shortest distance between two points; but the best definition should make use of the primary property and not a property which, however elementary, is a consequence of the primary one. Thus, let C be a point not on the line AB, and CD the perpendicular from C to AB. We may go from A to B in the direction AB, or we may go from A to B through C. In the latter case we make the same amount of progress as before in the direction AB, and make besides a progress in the independent direction DC, and back again: hence the path in the single direction AB is clearly seen to be shorter. Another definition is given by a recent ingenious writer, who says, "A straight line is a line which pierces space evenly, so that a piece of space from along one side of it will fit any side of any other portion." But what can the word "evenly," used with definite precision, mean if not "in one direction"? And if used with precision what is the use of the explanatory clause following it? The paper also alludes to broken lines, curves, parallels, etc., and concludes that there is no sufficient reason for rejecting the old familiar term "direction" which tells "which way," and with precision in the denominate numbers of angular measure, as "distance" tells "how far," and with precision in the denominate numbers of linear measure.

No. 8 will probably be printed in full in the BULLETIN. A summary is not at present available.

No. 9 is connected with a paper, recently presented to the

London Mathematical Society, which determines the order, generators and structure of the linear homogeneous groups in the Galois field of order p^n that have the quadratic invariants

$$\sum_{i=1}^{m} \xi_i \eta_i, \quad \xi_1 + \tau_1 + \sum_{i=1}^{m} \xi_i \eta_i.$$

For the special case $p^n = 2$, these groups are the first and second hypoabelian groups of Jordan. The present paper discusses the quaternary group G leaving invariant

$$\xi_1 + \eta_1 + \xi_1 \eta_1 + \xi_2 \eta_2.$$

For $p > 2$, every substitution of G of determinant $+1$ can be given the form

$$\begin{pmatrix} \nu\alpha\delta & -\nu\beta\gamma & \nu\alpha\beta & \nu\gamma\delta \\ -\nu\beta\gamma & \nu\alpha\delta & -\nu\alpha\beta & -\nu\gamma\delta \\ \nu\alpha\gamma & -\nu\alpha\gamma & \nu\alpha^2 & \nu\gamma^2 \\ \nu\beta\delta & -\nu\beta\delta & \nu\beta^2 & \nu\delta^2 \end{pmatrix} \equiv \begin{bmatrix} \alpha\gamma \\ \beta\delta \end{bmatrix}_\nu,$$

where $\nu(\alpha\delta - \beta\gamma) = 1$. Two such substitutions compound as follows:

$$\begin{bmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{bmatrix}_{\nu'} \cdot \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}_\nu = \begin{bmatrix} \alpha'\alpha + \beta'\gamma & \gamma'\alpha + \delta'\gamma \\ \alpha'\beta + \beta'\delta & \gamma'\beta + \delta'\delta \end{bmatrix}_{\nu'\nu}.$$

Hence those substitutions in which ν is a square (which may be supposed to be unity) form a quaternary linear group simply isomorphic to the simple group of linear fractional substitutions of determinant $\alpha\delta - \beta\gamma = 1$. Hence G has the factors of composition 2, 2, $\frac{1}{2}p^n(p^{2n} - 1)$. A like result holds for the ternary orthogonal group [see the BULLETIN for May, 1898, p. 388]. For $p = 2$, $n > 1$, the group G has a subgroup of index two simply isomorphic to the linear fractional group in the $GF[2^n]$, so that G has the factors of composition 2, $2^n(2^{2n} - 1)$.

No. 10 points out some connections which exist between various particular solutions of Laplace's equation, and shows how these may, in a certain sense, be regarded as a natural outgrowth of the harmonic functions with which the modern theory of functions has to deal. A few analogous equations are embraced in the two forms considered, the latter being

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \cdots = 0, \qquad (1)$$

$$\frac{\partial^2 U}{\partial x^2} - \left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \cdots \right) = 0. \qquad (2)$$

A number of theses or statements, self-evident or nearly so, are laid down at the outset; and the paper consists largely in their application. One thesis, for instance, is: Given any solution of (1) or (2) of the form

$$f_0(y, z, \cdots)\, \chi(x) + f_2(y, z, \cdots)\, \chi''(x) + f_4(y, z, \cdots)\, \chi\ (x) + \cdots,$$

and such that the entire coefficients of $\chi''(x)$, $\chi^{iv}(x)$, $\cdots$ in (1) or (2) are severally zero; then is

$$f_0(y, z, \cdots)\, \psi(x) + f_2(y, z, \cdots)\, \psi''(x) + f_4(y, z, \cdots)\, \psi^{iv}(x) + \cdots$$

a solution, ψ being an arbitrary function. *E. g.*, $1/r$ which is a solution of (1) may be thus expanded by the binomial theorem:

$$\frac{1}{r} = \frac{1}{x} - \frac{1}{2}\frac{y^2 + z^2}{x^3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{(y^2 + z^2)^2}{x^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{(y^2 + z^2)^3}{x^7} + \cdots.$$

If $\chi(x) \equiv \dfrac{1}{x}$, this may be written

$$\frac{1}{r} = \chi(x) - \frac{1}{2}\frac{1}{2!}(y^2 + z^2)\chi''(x) + \frac{1 \cdot 3}{2 \cdot 4}\frac{1}{4!}(y^2 + z^2)^2 \chi^{iv}(x) - \cdots.$$

Now let $\psi(x) \equiv \sinh x$, and we have

$$\sinh x \left[1 - \frac{1}{2^2}(y^2 + z^2) + \frac{1}{2^2 4^2}(y^2 + z^2)^2 - \cdots \right]$$

or $\sinh x J_0(\omega)$ as another solution of Laplace's equation. Again, if $\psi(x) \equiv x^n$, we have

$$x^n \left[1 - \frac{n(n-1)}{2^2}\frac{y^2 + z^2}{x^2} + \frac{n(n-1)(n-2)(n-3)}{2^2 4^2}\frac{(y^2 + z^2)^2}{x^4} - \cdots \right],$$

or, putting $x = r\mu$,

$$r^n \mu^n \left[1 - \frac{n(n-1)}{2^2}\frac{1 - \mu^2}{\mu^2} + \frac{n(n-1)(n-2)(n-3)}{2^2 4^2}\frac{(1 - \mu^2)^2}{\mu^4} - \cdots \right],$$

which is $r^n P_0(\mu)$, as another solution. In a similar manner, the solution $(y + \sqrt{-1}z)^n/r^{2n+1}$ gives, upon making

$$\chi(x) = 1/x^{2n+1} \text{ and } \psi(x) = \sinh x,$$

$$\sinh x(\cos n\varphi + \sqrt{-1}\sin n\varphi)J_n(\overline{\omega})$$

as a solution of (1). When concerned with (2), we may start with the solution $1/\sqrt{x^2 - (y^2 + z^2)}$, and proceed as before but making $\psi(x)$ equal to $\sin x$ or $\cos x$ instead of $\sinh x$. If $\varphi(ax + \beta y + \gamma z + \cdots)$ be expanded by Taylor's theorem in powers of $\beta y + \gamma z + \cdots$, (1) will be satisfied if $a^2 + \beta^2 + \gamma^2 + \cdots = 0$, and (2), if $a^2 - (\beta^2 + \gamma^2 + \delta^2) = 0$. Such expansion satisfies the requirements of the above-quoted thesis. By varying φ, a, β, γ, $\cdots$ a great number of particular solutions will be obtained.

$\varphi(ax + \beta y)$ where $a^2 + \beta^2 = 0$, or $\varphi(ax + iby)$ where $a^2 = b^2$ and $i^2 = -1$, is a solution of (1); this solution breaks up into two, the 1-terms and the i-terms. Similarly $\varphi(ax + \beta y)$ where $a^2 - \beta^2 = 0$, or $\varphi(ax + iby)$ where $a^2 = b^2$ and $\mathrm{i}^2 = +1$, is a solution of (2); this solution breaks up into two, the 1-terms and the i-terms. So $\varphi(ax + \beta y + \gamma z)$ where $a^2 + \beta^2 + \gamma^2 = 0$, or $\varphi(ax + iby + jcz)$ where $a^2 = b^2 + c^2$ and $i^2 = j^2 = -1$, is a solution of (1); the 1-, i-, j-, ij-parts of this solution are likewise solutions. Again, $\varphi(ax + \beta y + \gamma z)$ where $a^2 - (\beta^2 + \gamma^2) = 0$, or $\varphi(ax + iby + \mathrm{j}cz)$ where $a^2 = b^2 + c^2$ and $\mathrm{i}^2 = \mathrm{j}^2 = +1$, is a solution of (2) which can be broken up into four solutions. The expansion $\varphi(ax + \beta y + \gamma z + \delta\omega)$ breaks up into eight solutions of (1) if $a^2 + \beta^2 + \gamma^2 + \delta^2 = 0$, and of (2) if $a^2 - (\beta^2 + \gamma^2 + \delta^2) = 0$; in the first case they are the 1-, i-, j-, k-, ij-, ik-, jk-, ijk-parts, where $i^2 = j^2 = k^2 = -1$; and in the second, like parts where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = +1$. As an illustration, suppose $\varphi = \text{cosine}$, $a^2 = 1$, $b^2 = \frac{1}{2}$, $c^2 = \frac{1}{2}$; then the 1-terms in $\varphi(ax + iby + jcz)$ or

$$\cos\left(x + \frac{i}{\sqrt{2}}y + \frac{j}{\sqrt{2}}z\right)$$

give $\cos x \cosh \frac{y}{\sqrt{2}} \cosh \frac{z}{\sqrt{2}}$ as a solution of Laplace's equation. This when multiplied by the time factor denotes the velocity-potential of a standing wave in a V-shaped canal whose sides make angles of 45° with the vertical.

The object of No. 11 is to show the feasibility of constructing a machine such that if into it the hourly heights

of the sea be entered once, they will be distributed and added in the various ways required in tidal work. Nearly all of the many wheels used are of two kinds or patterns, a fact which would be of great importance should the construction of such a machine ever be attempted. A brief description of this analyzer will appear in Appendix No. 9, of the U. S. Coast and Geodetic Survey Report for 1897.

"Tidal Abacus" is a term applied in No. 12 to an apparatus consisting of a set of pairs of curves drawn upon movable boards together with a rack upon which they are mounted. The object of this contrivance is to enable a person to ascertain, by mechanical or graphic means, the height of the tide, or surface of the sea, at any given time past or future. The curves represent the harmonic components of the tide at the station for which they are drawn, and so the result of properly superposing them should agree with that obtained from computation or from a tide-predicting machine. A description of this apparatus is given in the Survey Report for 1894 (p. 183); but it was exhibited, for the first time, before the recent meeting at Boston. The tide represented by the curves is that at Port Townsend, Washington, where the amplitude of the diurnal wave often exceeds that of the semidiurnal.

The process outlined in No. 13 consists essentially in supplying by interpolation (without drawing the tide curve) the hourly heights of the semidiurnal portion of the curve. These heights are summed and analyzed for the semidiurnal components as if they were true heights of the surface of the sea. Knowing from observations the times and heights of the actual high and low waters and having all hourly heights of the semidiurnal tide supplied as just stated, the former heights when diminished by the latter give points on the diurnal curve. These are summed and analyzed for the diurnal components as if they were true heights of the surface of the sea. This method will be described in Appendix No. 9 of the Survey Report for 1897.

No. 14 showed how to determine upper and lower limits to the mass and moments of inertia of the earth's atmosphere. Expressing results in round numbers the investigation shows that the mass of the atmosphere must be greater than 1/1000000th and less than 1/1200th of the entire mass of the earth. Similarly the moments of inertia of the atmosphere must be greater than 1/555000th and less than

1/200th of the corresponding moments of the entire earth. This paper will be published in full in the *Astronomical Journal*.

No. 15 described two new forms of apparatus for measuring the acceleration of gravity. The first of these consists of a helical spring held rigidly at its upper end and vibrating vertically under the action of a mass attached at the lower end. Calling the mass of the spring m, the attached mass μ, the static elongation of the spring (under μ) ϵ, the periodic time of vibration τ, the acceleration of gravity is found to be

$$g = \epsilon \frac{\mu + \frac{1}{3}m}{\mu + \frac{1}{2}m} \left(\frac{2\pi}{\tau}\right)^2.$$

This formula ignores the effect of damping, of which, however, account may be easily taken if essential. It was pointed out that this apparatus and formula afford a useful class-room and laboratory method of measuring g. Usually, for such work, the ratio m/μ may be neglected, thus giving an approximation to g independently of μ. The other form of apparatus consists of a long piece of steel tape suspended vertically, being fastened rigidly at its upper end and carrying a relatively large mass at its lower end. The elastic pendulum thus formed will vibrate laterally and longitudinally. It can be shown that the effect of the latter vibration will be to produce a small modification of the lateral vibration for which correction can be applied, so that the value of g is given by a formula similar to that for a rigid pendulum. It was explained that both forms of apparatus may be enclosed in receivers from which the air may be exhausted and in which the temperature may be fixed by means of melting ice.

No. 16 called attention to the need of further researches to determine the gravitation constant and the mean density of the earth. It was shown that their product is not known with considerable precision and that we may hope to find their quotient. The importance of additional measurements of the gravitation constant was especially urged.

Mr. Hayford, in summarizing his useful paper (No. 17) states that his purpose is not to present anything new; but to point out a very weak point in tidal theory—to indicate where the most urgent tidal problem now lies. The height of the tide, referred to mean sea level, at any instant at any

given point may be represented accurately by an expression of the form

$$h = A_1 \cos(a_1 t + \beta_1) + A_2 \cos(a_2 t + \beta_2) \cdots$$

From pure theory, reasoning from the known motions of the moon, sun, and earth, and the Newtonian law of gravitation, it has been shown that if certain definite values be assigned to the quantities a_1, a_2, $\cdots$ (fixing the periods of the separate terms) each term then truly represents one of the invariable components of the tide. Here, after fixing merely the periods of the separate components the contribution of tidal theory ends, and the work of direct observation at the particular station under consideration begins. Theory does not furnish even a rough approximation for any station to the values of the A's, defining the amplitudes of the separate components, nor of the β's which fix the epochs. If we look at the actual tidal problem of nature it becomes evident that the reason why theory furnishes so little is that theory at present furnishes no adequate means of taking into account the effect of boundaries (bottom end shores) upon the tidal wave, although this effect is to modify the astronomical tide, computed for an ocean of uniform depth, so greatly that there is very little resemblance between the astronomical tide and the actual tide except that the periods of the separate components are identical in the two. The way in which the astronomical tide is modified by the boundaries may be indicated roughly as follows: The wave is everywhere a forced wave since the depth of the ocean is nowhere sufficient to allow the free wave to keep pace with the moon, and the great variations in depth make the lag much greater in some latitudes than in others; when one part of a wave outstrips another on account of the difference of depth a portion of the front of the wave becomes oblique and the wave acquires a northward or southward component in its motion which may ultimately cause different portions of the wave to coalesce; variations in depth produce radical changes in the amplitude of the wave—the amplitude being rapidly reduced by friction in small depths, and being greatly increased in passing from great to small depths (and decreased in the reverse case), because nearly the same amount of kinetic energy must be exhibited by a much smaller amount of water than before; one-quarter of the surface of the earth is land and the tidal wave is reflected to a greater or less extent at every contact with the shore and the direct wave is thus modified by combination with the reflected

wave. As a result of the extreme modifications due to the boundaries we find that the actual tidal wave, instead of progressing steadily westward around the earth once in each lunar day, fails even in the latitude in which it encounters least obstruction (just south of Australia, Africa, and South America) to make a complete circuit in even thirty-six hours; that the actual progress is to the northward rather than the westward over about one-third of the whole ocean surface (the whole Atlantic and a part of the Pacific); that the progress is to the eastward in at least two large areas (in the European Arctic ocean and near the west coast of South America); that the amplitude of the wave is continually varying during the progress of the wave between wide limits, viz., zero (nearly) and sixty feet (or more). The most important tidal problem before us at present is therefore the problem of determining the relation between the boundaries and the modifications produced by them upon the tidal wave. There is an abundance of facts at hand to be used in the investigation. This is a comparatively unworked portion of field of tidal investigation although considerable pioneer work has been done in it along purely mathematical lines. It would seem that the best chance of success lies in investigating this problem by a semi-graphic method based upon accurate charts of such a region as Chesapeake Bay in which the facts as to the tide are well determined at a large number of points and in which the direct effect of the moon and sun is small as compared with that of the wave transmitted from the ocean.

The full paper will be printed in *Science*.

The following is a summary of Mr. Preston's paper (No. 18): The geodetic operations in the United States executed by the Coast and Geodetic Survey fall naturally into three distinct periods. From 1807, when the work was authorized by Congress, to 1832, when it was resumed after having been delayed from various causes, may be regarded largely as an era of preparation and experiment. From 1832 to 1843 constitutes the second period, and was characterized by the beginning of work for the first time on an adequate geodetic scale. The third period comprises the period from the date of reorganization in 1843 to the present time. During the entire time between 1807 and to-day, taking away periods of inactivity occasioned by the Civil War, and other intervals when but little work was done, on account of lack of appropriations, we have fifty years of geodetic operations. During the existence of the Survey it has been three

times under the control of the Treasury Department, twice under the Navy, and once under law requiring its personnel to be Army or Navy officers.

The results of the work to the present time are the determination of 20,500 geographical positions, besides the execution of 38,500 square miles of topography, 100,000 miles of shore line, and 93,000 miles of deep sea sounding. 4,600 original topographic and hydrographic sheets are on file in the archives, from which considerable more than 1,000,000 charts have been made and distributed. There are 30,000 original volumes of observations. 203 base lines have been measured, of which 19 are primary, with an average length of 9,892 meters, and a probable error of 1/445,000 part. The three kinds of base-measuring apparatus have been, metallic bars, tape line, and a rod immersed in melting ice. The degrees of accuracy attained for the three years were, 1/1,000,000, 1/2,000,000, and 1/5,000,000, in the order stated. The arcs already measured consist of an oblique arc of 22 degrees on the Atlantic coast, and one from the Atlantic to the Pacific on the 39th parallel of latitude. The largest triangle yet observed has sides of 133, 167 and 190 miles. The highest station occupied is over 14,000 feet.

Besides the regular astronomical work, the Coast and Geodetic Survey has devoted considerable labor to the improvement of star catalogues, made necessary by refined observations in latitude, longitude and azimuth. The average probable error of a declination from the latitude lists may be given as less than one-fourth of a second, which degree of precision enables an observer to determine his latitude from twenty pairs in one evening with an uncertainty of only $\pm$ 10 feet. The Survey has sent out no less than 35 parties for the observation of solar eclipses, and transits of the inferior planets, and has determined the variations of latitude at three stations each one having been occupied more than a year. 5,000 miles of precise leveling have been executed, including four independent determinations of the height of St. Louis, two from the Atlantic and two from the Gulf of Mexico. Permanent magnetic observations have been in operation for five consecutive years at five distinct stations, at all of which, with one exception, self-registering apparatus has been continuously and exclusively employed. Gravity has been determined at 26 foreign stations, including points on all continents, and on many islands in the Atlantic and Pacific.

In physical hydrography work has been done in the exploration of the Gulf Stream, in the establishment of cotidal

lines, and in determinations of ocean depths from earthquake waves, besides the regular hydrographic work of the Survey. The practical results of the surveys are shown in the publication of annual reports, the issue of charts, notices to mariners, coast pilots, tide tables, etc.

Among the projects now in hand are the re-survey of Chesapeake Bay, the measure of an arc through the United States on the 98th meridian, a line of transcontinental levels, primary triangulation from San Francisco to the Mexican boundary, and hydrographic surveys along the Atlantic seaboard, on the Pacific coast, and at the mouth of the Yukon in Alaska. An extension of the great arc of the United States into Mexico and the British possessions has been proposed by Dr. Pritchett, the present Superintendent, and diplomatic representations between the interested governments looking towards concerted action in the near future have already been made.

In the ordinary prosecution of the field work about 50 parties are employed in the course of each year. Added to this, the purely hydrographic work is carried on by a fleet of 15 vessels, of which 8 are steamers.

Tidal indicators have been erected, or are in process of erection, at New York, Philadelphia and San Francisco, which furnish the navigator at any moment at a distance of one mile with necessary information as to the character and amount of the tide.

The Coast and Geodetic Survey is also charged with the establishment of trial speed courses for ships of the Navy, and the administration of an office of Standard Weights and Measures, from which prototypes are issued to the different States.

Nos. 19–37 are fairly well described by their titles, and are not well adapted for summarizing. Many of them will appear in the *Astronomical Journal* or in *Science*. Owing to the length of the programme some of the astronomical papers had to be presented in a separate room, thus giving an opportunity for the attendance of the members of the conference of astronomers and astrophysicists, which had been previously held at Harvard Observatory.

A noteworthy feature of the programme of Section A was a series of four admirable reports on the recent progress, and the present state, of certain branches of mathematical science, prepared by special invitation of the sectional committee. Each report occupied about an hour in reading,

and they were closely followed by large audiences at the last morning and afternoon sessions of the Section. The reports of Professors Brown and Webster were presented before a joint session of Sections A and B, the former being read by Professor R. S. Woodward (who was at the same time invited to present his own papers 14, 15, 16 to the joint session). Some of the reports will be printed in the BULLETIN and some in the *Proceedings*. Nos. 17, 18, 35, 36, 37 are also in the nature of reports of progress. The Council of the Association has now passed a resolution encouraging all the sectional committees to procure reports of this nature from competent scientists.

The Council elected to fellowship in the Association the following mathematicians and astronomers: Professor R. D. Bohannan, Mr. J. F. Hayford, Professor H. C. Lord, Dr. G. A. Miller, Miss Mary Proctor.

The next meeting of the Association will be held at Columbus, August, 1899, under the presidency of Professor Edward Orton. The officers elected for Section A are Dr. Alexander Macfarlane, Lehigh University, and Mr. J. F. Hayford, Coast and Geodetic Survey.

JAMES MCMAHON.

CORNELL UNIVERSITY.

NOTES.

THE fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held in room 35 of the Ryerson Physical Laboratory, University of Chicago, on Thursday and Friday, December 29th and 30th next. Titles, abstracts, and time requirements of papers to be presented should be in the hands of the secretary of the Section, Professor Thomas Holgate, Evanston, for the use of the programme committee, not later than December 10th.

AT the Bristol meeting of the British Association for the Advancement of Science, DR. MICHAEL FOSTER was elected president, PROFESSOR G. H. DARWIN one of the vice-presidents, and MAJOR P. A. MACMAHON a member of the Council. Grants of money to the extent of about seventy five hundred dollars were appropriated to purposes of scientific research, the section of mathematics receiving one-third of the amount. The following papers were presented before

the mathematical section :—By Lord KELVIN: "Graphic representation of the two simplest cases of a single wave."—By Professor EDGEWORTH : "The mathematical representation of statistics."—By Professor J. G. STOKES : "The imaginary of logic."—By Dr. JOHNSTONE STONEY : "The dynamical explanation of certain observed phenomena of meteor streams."—By Dr. JOHNSTONE STONEY : "A survey of that part of the scale upon which nature works, about which man has some information."—By Mr. J. H. VINCENT : "On the use of logarithmic co-ordinates."—By Professor HELE-SHAW : "A new method of describing cycloidal and other curves."—By Mr. E. T. WHITTAKER : "The recent history of the theory of the functions used in analysis."—By Colonel ALLAN CUNNINGHAM :—Report of the committee appointed for calculating certain mathematical functions, announcing the completion of the manuscript of a set of tables giving the residues of the powers of 2 for all prime moduli less than 1000. At future annual meetings of the mathematical section it is proposed to have presented a number of reviews of recent progress in various branches of pure mathematics.

THE Royal Scientific Society of Göttingen proposes as its prize problem for the year 1901 : To develop the law of reciprocity of the lth power residues for an arbitrary numerical body, l being an odd prime number. The statement of the problem is accompanied by the following explanation : Let l be an odd prime number, ζ one of the lth roots of unity other than 1, and k an arbitrary algebraic numerical body which contains the number ζ. If then ν, μ are any two integers of the body k, and w any prime ideal in k, the most general law of reciprocity for the lth power residues in the numerical body k is represented by the equation

$$\prod_{(w)} \left\{ \frac{\nu, \mu}{w} \right\} = 1.$$

In this formula the product extends over all prime ideals w of the body k, and the symbol $\left\{ \frac{\nu, \mu}{w} \right\}$ designates an lth root of unity to be suitably defined and unequivocally determined by the numbers ν, μ and the prime ideal w.* This law of reciprocity is to be worked out completely ; it is also to be demonstrated, at least for certain remarkable special cases

* See D. Hilbert, Theorie der algebraischen Zahlkörper, part 5, chapters 28–36, *Berichte der Deutschen Mathematiker-Vereinigung*, 1897.

or under appropriate simplifying assumptions. Particular value will be attached to the construction of numerical examples which appear capable of explaining and confirming the given law of reciprocity. The prize is one thousand marks. Manuscripts conforming to the usual regulations of such competitions will be received until February 1st, 1901.

HARVARD UNIVERSITY. The courses which were to have been given by Professor J. M. PEIRCE during the current academic year will be omitted as Professor Peirce has just been granted a year's leave of absence. In order to fill in part the gap thus made in the scheme of mathematical courses Professor F. S. WOODS of the Massachusetts Institute of Technology has been appointed lecturer in mathematics for the year 1898–99 to give a course in higher geometry which is to consist of three lectures a week throughout the year.

DURING the winter semester 1898–99, the several universities mentioned below offer the following mathematical courses:

UNIVERSITY OF KÖNIGSBERG. By Professor VOLKMANN: Electromagnetic theory of light, four hours; Seminar in mathematical physics, one hour.—By Professor STRUVE: Theory of satellites, two hours.—By Professor HÖLDER: Partial differential equations, two hours; Theory of numbers, four hours; Seminar, one hour.—By Professor MEYER: Critical pedagogical study of elementary mathematics, one hour; Integral calculus, three hours; Application of the integral calculus to geometry, one hour; Introduction to the mathematical theory of potential, two hours; Exercises in mathematical seminar, one hour.—By Professor SAALSCHÜTZ: Higher differential coefficients and reversion of series, one hour; The barycentric calculus of Möbius, one hour; Exercises in algebraic analysis, one hour; Determinants and other branches of algebraic analysis, four hours.—By Dr. RATHS: Spherical astronomy, three hours.—By Dr. COHN: Method of least squares, two hours; Theory of astronomical instruments, two hours.—By Dr. VAHLEN: Analytical geometry of space, four hours.

UNIVERSITY OF LEIPZIG. Professor SCHEIBNER will lecture, but the subjects of his courses are unannounced.—By Professor NEUMANN: Selected chapters of analytical mechanics, four hours; Mathematical seminar, one hour.—By Professor BRUNS: Celestial mechanics, four hours; Theory of probability, two hours; Seminar for scientific computing, two hours.

—By Professor MAYER: Introduction to analytical mechanics, calculus of variations, one hour.—By Professor ENGEL: Infinitesimal calculus, four hours; Theory of analytical functions, two hours; Seminar, one hour.—By Dr. HAUSDORFF: Projective geometry, four hours; Political arithmetic, three hours.

UNIVERSITY OF MUNICH. By Professor BAUER: Theory of equations; Surfaces of the third order; Mathematical seminar.—By Professor LINDEMANN: Theory of functions of a complex variable; Application of the infinitesimal calculus to the theory of curves and surfaces; The problem of the quadrature of the circle; Mathematical seminar.—By Professor SEELIGER: Theory of potential and the figure of the heavenly bodies; Astronomical colloquium.—By Professor PRINGSHEIM: Differential calculus; Exercises on the former; Theory of numbers.—By Professor GRAETZ: Analytical mechanics; Exercises in mechanics.—By Dr. DONLE: Introduction to the electro-magnetic theory of light.—By Dr. BRUNN: Exercises connected with the reading of mathematical classics.—By Dr. DOHLEMANN: Descriptive geometry, with exercises; Graphical statics, with exercises.—By Dr v. WEBER: Introduction to analysis; Ordinary differential equations.

UNIVERSITY OF STRASSBURG. By Professor REYE: Geometry of position, three hours; Analytical mechanics, two hours.—By Professor BECKER: Selected chapters of celestial mechanics, two hours.—By Professor WEBER: Theory of elliptic functions, four hours; Theory of algebraic numbers, two hours; Advanced seminar, in conjunction with Dr. Wellstein, one hour; Lower seminar, one hour.—By Professor ROTH: Algebraic analysis and determinants, three hours; Analytical geometry of space, two hours; Ordinary differential equations, two hours.—By Professor KRAZER: Infinitesimal calculus, four hours; Analytical geometry of the plane, three hours; Exercises in the infinitesimal calculus, two hours.—By Dr. WELLSTEIN: Invariant theory of binary forms.

WE learn from a recent number of *Nature* that DR. G. VAILATI, writing in the *Bolletino di Storia e Bibliografia Matematica*, has brought to light an obsolete book of Euclid dealing with balances and the principle of the lever. Euclid's reasoning seems to have been based on two axioms: 1° that if a loaded lamina balances about a horizontal axis, it will continue to balance when the weights are displaced parallel to the axis; 2° if a lamina balances horizontally about two

intersecting axes in its plane, it will also balance about their point of intersection; he arrives at the conditions of equilibrium of a lever whose arms are in the ratio of two whole numbers by a method closely analogous to that adopted by Archimedes. The work has become known through an Arabic translation by Ibn Musa in the National Library at Paris, an account of which was given in 1851 by Woepke in the *Journal Asiatique*, but seems to have been overlooked by mathematicians.

THE papers and proceedings of the first International Congress of Mathematicians held at Zürich, August 9–11, 1897, have just been issued in one volume by Teubner under the editorship of PROFESSOR FERDINAND RUDIO. From the press of the same publisher has recently appeared the first part of the first volume of the "Encyclopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen" (see BULLETIN, 2d Series, vol. 3, p. 356, vol. 4, p. 32) projected by the academies of science of Göttingen, Munich, and Vienna under the immediate editorship of PROFESSORS H. BURKHARDT, of Zürich, and W. F. MEYER, of Königsberg.

THE works of LAGUERRE are being published under the auspices of the Paris Academy of Sciences and the editorial supervision of PROFESSORS HERMITE, POINCARÉ, and ROUCHÉ, in an edition of two volumes to be occupied respectively with analysis and geometry; Gauthier-Villars announces that the first volume is ready. Of the publications of the same firm, the fourth and concluding volume of TANNERY and MOLK'S "Éléments de la théorie des fonctions elliptiques" is in type, and the fourth and last part of MÉRAY'S "Leçons nouvelles sur l'analyse infinitésimale et ses applications" has just been received from the press. The former of these two volumes is devoted to the integral calculus and applications, the latter to the classic geometrical applications of the infinitesimal calculus.

THE MACMILLAN COMPANY announces the appearance of an "Introduction to the theory of analytic functions," by PROFESSORS JAMES HARKNESS, of Bryn Mawr College, and FRANK MORLEY, of Haverford College; the first volume of a "Treatise on infinitesimal analysis," by PROFESSOR W. B. SMITH, of Tulane University; a "Text-book on the calculus," by PROFESSOR P. A. LAMBERT, of Lehigh University; and a "Treatise on graphics," by PROFESSOR F. N. WILLSON, of Princeton University.

PROFESSOR F. MORLEY has resumed the chair of mathematics at Haverford College after a year abroad on leave of absence.

ASSISTANT PROFESSOR S. I. BAILEY has been promoted to an associate professorship of astronomy at Harvard University.

DR. E. P. CHILDS has been appointed professor of physics in the University of New Mexico.

MR. J. M. POOR has been appointed instructor in astronomy in Dartmouth College.

PROFESSOR W. C. TINDALL, until recently head professor of the department of mathematics in the Missouri State University, died on September 17th.

THE University of Pennsylvania has given its alumnæ mathematical fellowship for women to MISS R. H. VIVIAN.

FATHER CYPRIEN, of the monastery of Mount Athos, formerly the well-known explorer, Prince C. Wiasemsky, contributes to the *Revue générale des Sciences* for the 15th of September, 1898, an article entitled: "New theory of transcribed spheres."

PROFESSOR H. POINCARÉ, of the University of Paris, contributes an article "On the foundations of geometry" to the October number of the *Monist*.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADAM (C.). See DESCARTES.

ALLEN (J.). See TANNER (J. H.).

BEEUER (A.). Elementar entwickelte Theorie und Praxis der Functionen einer complexen Variabelen in organischer Verbindung mit der Geometrie. Vienna, Daberkow, 1898. 8vo. 8 and 205 pp.

DESCARTES. Oeuvres de Descartes. Publiées par Charles Adam et Paul Tannery, sous les auspices du ministère de l'instruction publique. Correspondance, II (mars 1638–décembre 1639). Paris, Cerf, 1898. 4to. 23 and 657 pp.

FATTOR (L.). Di una construzione delle coniche per punti e per tangenti. Cividale, Fulvio, 1898. 8vo. 7 pp.

GIUDICE (F.). Nozioni sulle transformazioni puntuali e sui gruppi continui. Brescia, Appolonio, 1898. 8vo. 10 and 144 pp.

GOETTLER (J.). Conforme Abbildung eines von confocalen elliptischen und hyperbolischen Kurven nter Ordnung begrenzten Flächenstückes auf der Halbebene. (Programme.) Passau, Waldbauer, 1898. 8vo. 34 pp. M. 1.00

JAHN (F.). Ueber die Entwicklung der doppelt periodischen Funktionen in doppelt unendliche Fourier'sche Reihe. (Diss.) Strassburg. 4to. 55 pp.

KEMMER (G.). Ueber die Verwandlung von Projectivitäten in Involutionen und von Reciprocitäten im Polarsysteme durch Anwendung von Projectivitäten. Darmstadt, 1898. 8vo. 24 pp. M. 1.00

KOVALEVSKY (S.). Sophie Kovalevsky, ein weiblicher Professor, von J. v. Zednik. Prague, 1898. 8vo. 15 pp. M. 0.20

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